impact math focus on the future

Number Sense & Numeration

Ontario Ministry of Education and Training
Queen's Park, Mowat Block,
Toronto, Ontario
M7A 1L2

Impact Math is a professional development program to help teachers of Grades 7/8 implement the new Mathematics curriculum. The program was developed by the Impact Math team at the Ontario Institute for Studies in Education of the University of Toronto (OISE/UT). The development of this resource document was funded by the Ontario Ministry of Education and Training. This document reflects the views of the developers and not necessarily those of the Ministry.

Impact Math Team:

Eleanor Adam
Manager, Program Development, Continuing Education, OISE/UT

Lorne Cook
Director, Impact Math

Brendan Kelly
Professor, OISE/UT

Doug McDougall
Professor, OISE/UT

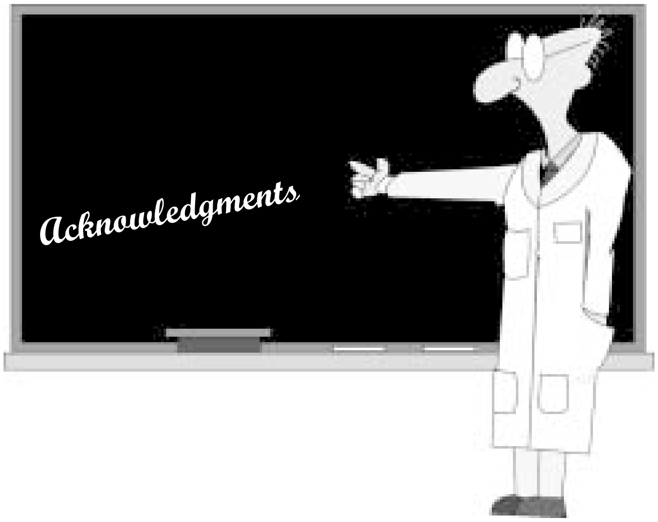
Joanne Quinn
Director, Continuing Education, OISE/UT

Carol Rolheiser
Associate Chair, Teaching and Learning Department, OISE/UT

Author:

Brendan Kelly

Illustrator:
Taisa Dorney



This module is the final product in a series of drafts, revisions and field tests conducted during 1999. Enhancing the usefulness of this booklet is the plethora of wonderful samples of student work that appear under the heading "WHAT YOU MIGHT SEE." For these samples we are deeply indebted to the Grade 7 and 8 students of the following teachers:

West Humber Junior Middle School Toronto District School Board	Don Mills Middle School Toronto District School Board	Norseman Junior Middle School Toronto District School Board
Erin Altosaar	Graham Beer Marcia Schantz	Jason Johnston

We wish to thank these excellent teachers and their students for their invaluable contribution to this resource. Mike Wierzba and Barry Scully served as the liaisons who conducted the field tests, gathering information from students and teachers and providing insightful suggestions regarding the scoring guides, the activities, and the content of this module.

We acknowledge a substantial debt to the following members of the *Impact Math* cadre of trainers who made important suggestions for improvements to the activities and the scoring guides to facilitate their use by teachers and strengthen their use as aids to report card preparation (see pp. 94–95).

Katie Branovacki	Trevor Brown	Jeri Lunney	André Mailloux	Pat Margerm
Lee McMenemy	Laura Romano	Caroline Rosenbloom	Jan Scully	Emilia Veltri

In addition to their contribution noted above, Trevor Brown and Lee McMenemy for this module in particular, suggested some modifications to the activities that have significantly enhanced the final product.

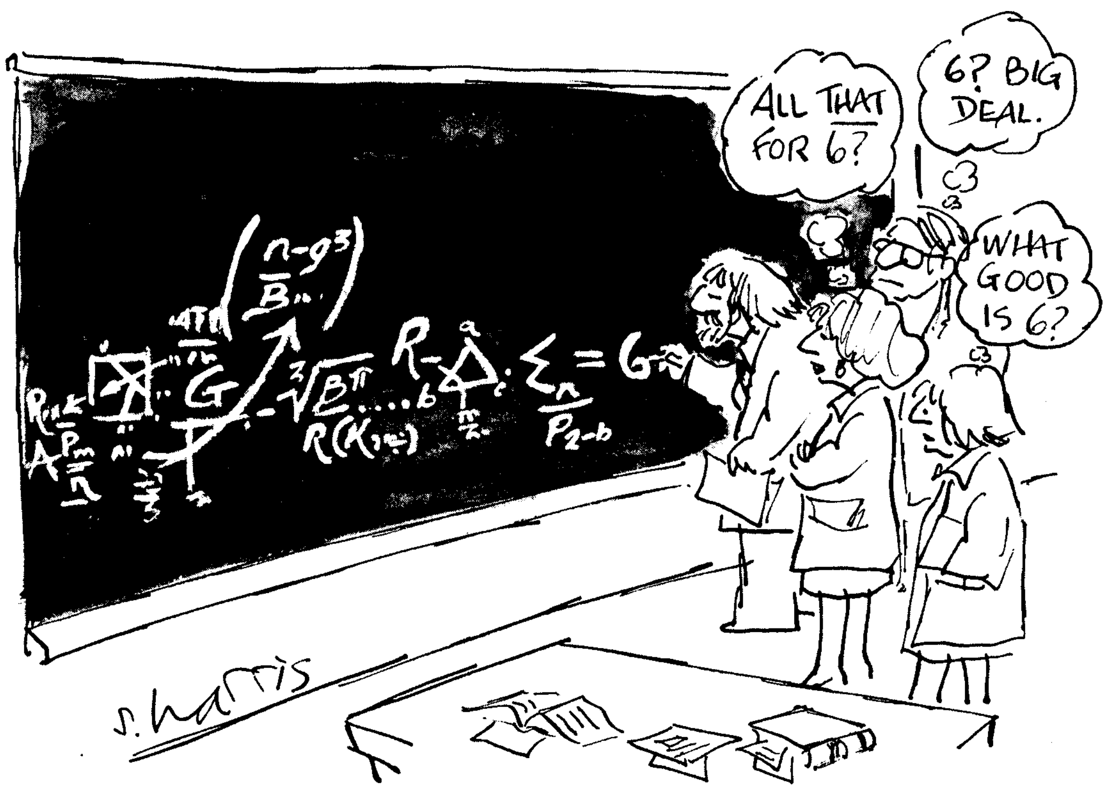
A special debt of gratitude is owed to friend and colleague Ed Barbeau of the Mathematics Department at the University of Toronto whose useful suggestions and encyclopedic knowledge have embellished several modules. In the Geometry module, in particular, he provided the reference to Fagnano's Theorem and suggested that we encourage exceptional students to apply their knowledge of algebra to proving Euler's Theorem for prisms and pyramids (see *Geometry & Spatial Sense* p. 37).

The creation of such a document as this involves many stages of revision, rewriting, and reorganization, carrying with it a multitude of opportunities for errors. We are eternally grateful to Rosemary Tanner's sharp eye and expert editorial skills for the many errors and omissions that she purged from the manuscript at each stage. Rosemary also raised several queries that challenged us to double-check our research and our sources.

Contents

PART I – PHILOSOPHY & RATIONALE

PART II – WHAT'S NEW IN NUMBER SENSE & NUMERATION?



PART III– NUMBER SENSE & NUMERATION IN GRADE 7

PART IV– NUMBER SENSE & NUMERATION IN GRADE 8

Introduction to the Modules

The Ontario Curriculum, Grades 1– 8: Mathematics, issued in 1997, has redefined the elementary school mathematics curriculum for Ontario. New expectations for student learning require the teaching of new mathematical topics as well as a shift in emphasis of content previously taught. In particular, the new document reflects the growing need for students to expand their skills in processing information, managing data, problem solving, and using technology to achieve these ends. While there is a reduced attention to rehearsing rote skills, such as long division with large divisors or extraction of roots by the formal method, there is a reaffirmation of the need for students to master the multiplication tables and fundamental pencil-and-paper skills that underpin arithmetic facility. Such skills are intended to support the intelligent use of technology in performing complex computations of the type that arise in so-called "real-world" contexts.

Implicit in this document is the demand for new or revised methods of instruction and assessment. Educational research of the past twenty years has mounted a compelling argument for a knowledge-building approach to instruction (see page 9) that reduces the role of the teacher as purveyor of information and enhances the teacher's role as facilitator of learning. With this shift in instructional methodology comes a corresponding demand for change in methods of assessment (see pp. 10 – 12).

The call for such changes in curriculum, instruction, and assessment has created a need for teachers of grades seven and eight to plan new programs in mathematics from the plethora of print and electronic resources currently available. Since most of these teachers are responsible for many subject areas in addition to mathematics, the consolidation of these materials into a set of coherent lessons is daunting. To support teachers in this quest, the Ministry of Education and Training has commissioned a set of five modules (of which this is one) that gather together many of the extant resources in a single reference package. Each module addresses one of the five strands in the new curriculum.

Though they address different content strands, all modules have the same format. Part I outlines the rationale underpinning the ideas and activities developed in the module. Part II provides a brief instruction for teachers on the new content or approaches in that strand. Part III provides a set of four sample activities for Grade 7. (Included in each activity is a 2-page template identified as *Student Pages*. To avoid excessive photocopying, it is recommended that you make one class set of these materials and laminate them. Then distribute to the students as needed, having them record responses in their notebooks.) Together these four activities constitute an authentic task designed to consolidate and extend earlier developmental activities. This unit is intended to model the instructional and assessment philosophies discussed in Part I. **It is not intended to cover the entire content of the strand, nor to replace any resources presently used, but rather to supplement the current program.** Included in Part III under the heading "What You Might See" are samples of student work, classified by achievement level, and presented opposite a rubric that will help you assess the work of your students. Part IV parallels Part III, except it is keyed to the Grade 8 unit. However, it is recommended that all teachers familiarize themselves with the contents of both Parts III and IV. Part IV concludes with a selected list of appropriate print and media resources at the Grade 7–8 levels and some useful Internet addresses to fulfill the intent that the module provide a single reference to help teachers implement the new curriculum.

A Letter To the Teacher

Dear colleague,

This letter is written as the final module, *Number Sense and Numeration,* is about to go to press. It is my wish, as author, to share with you some final throughts as my work with teachers and students in the field testing of these modules comes to an end.

During the course of this project, I have learned (or perhaps rediscovered) two principles that have emerged from my observations of the work of teachers and students in Ontario.

- When a class is deeply involved in a mathematical investigation, the invisible stimulus is almost inevitably a capable, enthusiastic, and caring teacher.
- When provided with sufficient support, most children not only enjoy being challenged (see student comments on p. 56), but they often respond at levels beyond what we might ever imagine.

This module, even more than the others, was developed with the latter principle foremost in mind. It represents a significant departure in process and depth from what has been traditionally expected of students at this level. While following the content guidelines, the activities challenge students in grades seven and eight to reach greater depths in their thinking, to explore openly, and to discuss their ideas. Although some routine exercises are included, there is an attempt to move beyond the rehearsal of short algorithmic applications to deeper and richer investigations that foster the development of fundamental concepts.

The sample units in this module contain independent activities to enable you to use them separately at various times throughout the school year. We suggest you apply Fullan's[1] maxim, "Think big – start small," and try only one or two of them in your first year, expanding further as you become more comfortable with this approach.

You will find that some students will rise to the challenge and announce with joy their newly-awakened interest in mathematics. Others may shrink away initially and attempt to opt out. When difficulties arise, you will need to provide support with manipulative materials and informal discussions of the ideas involved. Grouping for instruction using formats such as Think-Pair-Share is an excellent way to offer peer group support for those experiencing difficulties.

The work collected in the field testing of these modules attests to the remarkable resourcefulness and insights of students in a stimulating learning environment. Creating such a working environment for all students is a daunting task, and I humbly express my admiration for those teachers who are successful in this quest.

Thank you for your effort and professionalism. The students you teach will acquire the mathematical literacy they need to perform well in the 21st century and for that all members of our society owe you a debt of gratitude.

Brendan Kelly

[1]Fullan, Michael. *Change Forces*. London, U.K.: Falmer Press, 1993.

The Rationale for Number Sense & Numeration

Number Sense

The rationale for the *Number Sense and Numeration* strand in the Ontario mathematics curriculum is given on pages 10 and 11 of *The Ontario Curriculum, Grades 1- 8: Mathematics.* This rationale clarifies what is intended under the heading "number sense:"

> *Students also need to develop "number sense." Number sense includes:*
>
> - *an appreciation of and ability to make quick order-of-magnitude approximations with emphasis on quick and accurate estimates in computation and measurement;*
> - *the ability to detect arithmetic errors;*
> - *knowledge of place value and the effects of arithmetic operations;*
> - *a grasp of ideas about the role of numbers and about their multiple relationships;*
> - *an appreciation of the need for numbers beyond whole numbers.*

Part III of this module introduces Fermi problems (see p. 16 for a definition) as a vehicle for developing number sense. Students are presented with sets of open-ended problems that require organized estimation procedures involving order-of-magnitude approximations. In particular, students are asked to estimate how many times their heart will beat in their lifetime and to compare this with the capability of an artificial heart. Another activity in that sample unit engages students in estimating the volume of the *Spaceship Earth* pavillion at the *EPCOT Center* and deciding whether all the golf balls in the world today could be contained in that geodesic dome.

Mental Mathematics & Estimation

The rationale for the *Number Sense and Numeration* strand also asserts:

> *By the end of Grade 6, students should have consolidated their understanding of basic computational facts and be able to use computational strategies to do mental mathematics … Techniques of mental mathematics should be introduced along with concepts of place value and the use of pencil-and-paper calculations.*

In the sample unit presented in Part IV of this document, students are prompted to develop "mental math" algorithms for calculating tips and sales tax. Different algorithms are compared for efficiency and ease of execution. Another activity in that unit has students compare fractional approximations to π used by various cultures through the ages. This is followed by an activity that engages students in discussions aimed at resolving some classical fraction paradoxes. These learning experiences are intended to address the requirement that "Mathematics instruction should help students gain conceptual understanding as well as use fractions and rational numbers effectively and accurately."

The Role of Technology in the Number Sense & Numeration Strand

The policy on the use of technology, as embodied in *The Ontario Curriculum, Grades 1- 8: Mathematics,* is stated on page 7 of that document:

> *Students are expected to use calculators or computers to perform operations that are lengthier or more complex than those covered by the pencil-and-paper expectations. When students use calculators and computers to perform operations, they are expected to apply their mental computation and estimation skills in predicting and checking answers. Students will also use calculators and computers in various experimental ways to explore number patterns and to extend problem solving.*

More detailed statements pertaining to the *Number Sense and Numeration* strand are given on page 11.

Calculators

> *The ability to use calculators intelligently is an integral part of number sense. It should be noted that the use of calculators does not do away with the necessity for students to master the fundamental mathematical operations. Students should use calculators in their schoolwork, just as adults use calculators for many purposes in the course of their daily lives. More importantly, students must learn when it is appropriate to use a calculator and when it is not. They must learn from experience with calculators when to estimate and when to seek an exact answer, and how to estimate answers to verify the plausibility of calculator results. Calculators allow teachers to engage students in meaningful mathematical investigations, such as solving science problems with large numbers, before their skill with pencil-and-paper computation is equal to the task. Proper calculator use stimulates the growth of number sense in students.*

Computers

> *The computer is an important tool used by mathematicians to perform a wide variety of tasks; the ability to use computers effectively and appropriately is central to students' development of mathematical competence.*
>
> *An important use of computer software is to engage students in the exploration of concepts. Computer programs should help students develop number sense and deal with large amounts of data in an organized way. Spreadsheets should be used by all students to manage and operate on long lists of numbers. Also, the computer can serve as an aid to students in clarifying operations rules that will help them develop concepts used in early algebra.*

The Grade 7 sample unit in Part III requires the judicious choice of pencil-and-paper or calculator computation. Some order-of-magnitude problems can be done mentally by the manipulation of exponents of numbers expressed in scientific notation. For other problems, such as the conversion of seconds to years, the calculator is the more appropriate tool. In the Grade 8 sample unit in Part IV, mental computation is encouraged for the calculation of sales tax and tips but the calculator is the preferred tool in the comparison of various fractional approximants to π. In Activity 4 of that unit, students who have access to a graphing calculator are encouraged to run the program PASCAL to display the first 64 rows of Pascal's triangle and to determine the percent of even numbers in that array.

Understanding the Learning Process & Its Impact on Instruction

In this and the other four modules, we present activities that attempt to incorporate a range of instructional approaches. The students are sometimes given information and required to read, interpret, and apply it in an exercise. In other cases, the students must investigate, explore, and discover concepts that lurk beneath the surface of an activity. In some cases, the students work individually, while in others they work collaboratively or cooperatively. For example, in Activity 1 of Part III of this module, students collaborate in the creation of a problem bank consisting of CN Tower word problems. The students use data about the CN Tower, its elevators, its Sky Pod, its stairs, and its Space Deck to create word problems that a fellow student must solve. The word problems are then edited by another student and the teacher who compiles them in a class problem bank for future use on tests or quizzes. In several activities of this unit, students are referred to Internet sites to obtain information and to verify calculations. Activity 3 in Part IV involves students in mathematical dialogues about some classical fraction paradoxes. Different interpretations and points of view are presented and defended and fraction concepts are discussed first in small groups and then with the entire class. In Activity 4 of that unit, students complete the first sixteen rows of Pascal’s triangle and then conjecture about the percent of even numbers in Pascal’s triangle as the number of rows is increased. By shading cells in a template showing the first 32 rows of Pascal’s triangle (see template ***The Pascal Parity Game***, page 92), students test their conjectures. Those with access to graphing calculators can run the program PASCAL (see page 93) to test their conjectures for larger numbers of rows of Pascal’s triangle.

In view of the multiple perspectives on how children learn, one might assume that all traditional approaches to teaching will disappear as these philosophies are incorporated. However a response to the question “What should I see in a [NCTM] Standards-based mathematics classroom?” the *NCTM 1997–98 Handbook* presents a balanced and accessible image of effective instruction:

> *First and foremost, you'll see students doing mathematics. But you'll see more than just students completing worksheets. You'll see students interact with one another, use other resources along with textbooks, apply mathematics to real-world problems, and develop strategies to solve complex problems.*
>
> *Teachers still teach. The teacher will pose problems, ask questions that build on students' thinking, and encourage students to explore different solutions. The classroom will have various mathematical and technological tools (such as calculators, computers, and math manipulatives) available for students to use when appropriate. The teacher may move among the students to understand their thinking and how it is reflected in their work, often challenging them to engage in deeper mathematical thinking.*

Assessment: Rubrics & Achievement Levels

The changes in curriculum and instruction described on the preceeding pages have significant implications for assessment and evaluation. Among these implications is the shift from norm-referenced to criterion-referenced assessment, as described on page 1 of *The Assessment Standards for School Mathematics* published by the NCTM in 1995:

> *At present, a new approach to assessment is evolving in many schools and classrooms. Instead of assuming that the purpose of assessment is to rank students on a particular trait, the new approach assumes that high public expectations can be set that every student can strive for and achieve, that different performances can and will meet agreed-on expectations, and that teachers can be fair and consistent judges of diverse student performances.*

The Ontario Curriculum, Grades 1–8: Mathematics (see pp. 4–5) also embraces the move to criterion-referenced assessment and includes four levels of achievement for describing student performance:

> *High achievement is the goal for all students, and teachers, students, and parents need to work together to help students meet the expectations specified. The achievement levels are brief descriptions of four possible levels of student achievement. These descriptions, which are used along with more traditional indicators like letter grades and percentage marks, are among a number of tools that teachers will use to assess students' learning. The achievement levels for mathematics focus on four categories of skills: problem solving, understanding of concepts, application of mathematical procedures, and communication of required knowledge. When teachers use the achievement levels in reporting to parents and speaking with students, they can discuss with them what is required for students to achieve the expectations set for their grade.*

Descriptions of the four levels of achievement for problem solving, concepts, applications, and communication are shown on page 9 of that document. These are the levels for concept understanding:

knowledge/skills	Level 1	Level 2	Level 3	Level 4
Understanding of concepts	**The student shows understanding of concepts:**			
	– with assistance	– independently	– independently	– independently
	– by giving partially complete but inappropriate explanations	– by giving appropriate but incomplete explanations	– by giving both appropriate and complete explanations	– by giving both appropriate and complete explanations and by showing that he or she can apply the concepts in a variety of contexts
	– using only a few of the required concepts	– using more than half the required concepts	– using most of the required concepts	– using all of the required concepts

A table such as the one above that describes levels of achievement is called a *rubric*. Included with the student activities, in this and the other modules, are rubrics and samples of student work that exemplify the levels of student performance as defined in *The Ontario Curriculum, Grades 1–8: Mathematics.*

The release of the first module in this series, *Data Management & Probability*, was met with widespread enthusiasm. It confirmed our belief that teachers need and want materials to help them implement the new mathematics curriculum. Of particular interest to teachers are the issues associated with assessment and evaluation. The shift in emphasis from rote learning to higher-order processes, such as problem solving, drawing inferences, and communicating mathematical conclusions, requires that methods of performance assessment be added to the battery of devices that teachers use to assess mathematical learning. As observed in the NCTM publication *Curriculum and Evaluation Standards for School Mathematics: Addendum Series – A Core Curriculum* (1992):

> *Questions eliciting open-ended responses require more holistic approaches for scoring. Indirectly, they convey to students the need to communicate their ideas clearly and to construct their responses for a purpose. The impact on the curriculum of this type of assessment is to hold students accountable for demonstrating their understanding of connected ideas rather than displaying their proficiency with disconnected skills.* (p. 11)

One of the most important devices for the holistic scoring of higher-order tasks is the rubric. The rubric shown on page 10 is an example of what is called a "general rubric." In its publication *Assessment Standards for School Mathematics* (1995), the NCTM defines a general rubric as "an outline for creating task-specific rubrics" (p. 90). Furthermore it defines a "task-specific rubric" as a rubric that "describes levels of performance for a particular complex task and guides the scoring of that task consistent with relevant performance standards." In this module we present, under the heading WHAT YOU MIGHT SEE, samples of student responses to the activities. Large samples of student work collected during the field tests of these materials were used to create scoring guides. These guides are task-specific rubrics. You will notice however that they evaluate the "product", i.e., the student work, while the general rubric shown on page 10 includes an observational component of assessment (e.g., "with assistance," "independently"). Since there can be no observational component in the assessment of *completed* student work, the scoring guides in this book do not use phrases such as "independently" or "with assistance." **It is expected that teachers will use each scoring guide as a starting point in the development of a task-specific rubric that will evolve as it is used with students.**

As a reminder that students should be assessed on more than just the "written product," we have included on p. 89 a "process-oriented" scoring guide that may serve as a starting point for creating your own rubric for an observation-based assessment of problem solving. As noted in the other modules, it is important to recognize that the creation of rubrics is highly subjective and is more an art than a science. It will probably take several drafts before you develop a rubric that works for you. In the *TIMSS Monograph #1: Curriculum Frameworks for Mathematics and Science* (1993), Robitaille et al. issue this caveat:

> *Measuring educational achievement is difficult from both a conceptual and a practical perspective. What counts as "achievement" is not always easy to discern and even when a concept of achievement has been clearly explicated, ways and means for assessing it are not easily devised. The ongoing debate about educational measurement and the increasing number of alternative assessment approaches proposed in educational circles attest to this problem.* (p. 36)

PART II

What's New in Number Sense and Numeration?

The Changing Role of Algorithms in the Mathematics Curriculum

The algorithm has been the cornerstone of elementary school mathematics programs ever since arithmetic was incorporated in the school curriculum. Named after Arab mathematician *Mohammed ibn-Musa al-Khowarizmi* (see the *Patterning & Algebra* module p. 40), an algorithm is merely a mathematical recipe for performing a particular mathematical operation or function. The procedures for multiplying two numbers, extracting a square root, or bisecting an angle with ruler and compasses are all examples of algorithms. Similarly, the set of instructions in a computer program that generates a fractal, solves an equation, or graphs a function is also an algorithm. Until recently, the main goal of elementary mathematics was to help students master standard algorithms for adding, subtracting, multiplying, and dividing whole numbers, decimals, and fractions. Mastery of these algorithms by pencil-and-paper computation was an important numeracy and survival skill in the pre-calculator era. Bank tellers, store keepers, and virtually all citizens required some facility with these algorithms merely to function on a daily basis.

The emergence of the hand-held calculator and the personal computer in the late 1970s challenged the central role of computational competence as the *raison d'être* of the elementary school mathematics curriculum. Many educators asked why it was necessary for humans to dedicate so much time to mastery of skills for which the new machines were so much better adapted. At the same time, it was recognized that some skill in pencil-and-paper and mental computation may be needed to judge whether the outputs of these machines were reasonable. In the decades that followed, there has been a gradual shift in the role of computational algorithms in the elementary school curriculum. The heavy emphasis on computational facility with pencil and paper has been somewhat reduced in favour of a greater attention to estimation, approximation, and order of magnitude judgements, collectively referred to as *number sense*. These gradual shifts in emphasis have been captured in the table below, presented in *Developing Number Sense* (p. 13) from the *Addenda Series Grades 5–8: Curriculum and Evaluation Standards for School Mathematics.*

> *The NCTM Curriculum and Evaluation Standards recommends a number of changes in the middle grades mathematics curriculum…[The table] highlights the areas that are to receive increased and decreased emphasis.*

Increased Attention	**Decreased Attention**
• Developing number sense	• Memorizing rules and algorithms
• Developing operation sense	• Practicing tedious paper-and-pencil cornputations
• Creating algorithms and procedures	• Finding exact forms of answers
• Using estimation both in solving problems and in checking the reasonableness of results	• Memorizing procedures, such as cross-multiplication, without understanding
• Exploring relationships among representations of, and operations on, whole numbers, fractions, decimals, integers, and rational numbers	• Practicing rounding numbers out of context
• Developing an understanding of ratio, proportion, and percent	• Developing skills out of context
	• Learning isolated topics

Traditional Algorithms vs. Student-Created Algorithms

Most mathematics educators agree that students should learn the multiplication table involving products up to 10×10 and should master some simple algorithms for performing arithmetic operations. (In fact, the emergence of computer science and with it an increased emphasis on what is termed *finite mathematics* has heightened the awareness of the importance of algorithms in mathematics.) However, there is a shift in the rationale for teaching algorithms – away from a focus on the skill and toward an understanding of the process.

> *Even common elementary school algorithms for arithmetic take on a new dimension when viewed from the perspective of contemporary mathematics: rather than stressing the mastery of specific algorithms–which are now carried out principally by calculators or computers–school mathematics can instead emphasize more fundamental attributes of algorithms (e.g., speed, efficiency, sensitivity) that are essential for intelligent use of mathematics in the computer age. Learning to think algorithmically builds contemporary mathematical literacy.* (Steen, p. 7.)

With the reduced need to develop skill in the rapid execution of an arithmetic operation, the requirement that students master *the* algorithm for addition is replaced by the expectation that students will learn *an* algorithm for addition. Proficiency in its execution becomes less important than an understanding of how and why it works. Hence, involving the student in the development of algorithms to perform arithmetic operations is becoming the preferred approach.

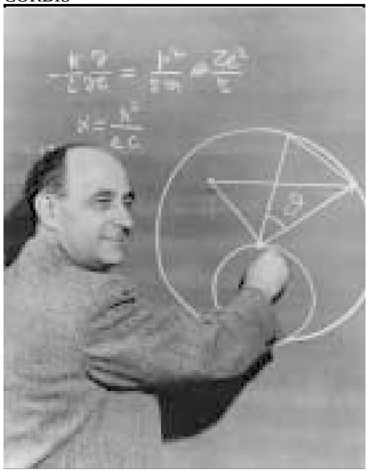
> *We need to teach students how to develop their own algorithms and break away from the traditional algorithm instruction. Students should identify their own steps and ways for solving problems. Teaching algorithms [in this way] helps kids to organize their thoughts and to see the importance of this organization.* (Mingus et al., p. 40.)

How Can Students Create Their Own Algorithms?

The NCTM's 1998 Yearbook, *The Teaching and Learning of Algorithms in School Mathematics*, (see reference under "Morrow" on p. 96 of this module), presents a variety of perspectives on algorithms and their place in the school mathematics curriculum. Several articles in that anthology present student-generated algorithms for whole number operations (e.g., see articles by Carroll, Ron, McClain et al., Baek, and Whitin & Whitin). There are also several articles that present student-generated algorithms for fraction operations (e.g., see articles by Huinker, Lappan, Sharp, and Brendefur & Pitingoro). In these articles, you will find a rich treasury of ideas that work in the classroom. The shift from the repetitive drill of rote skills to exploration, discovery, and communication of new algorithms is evident throughout. Also evident is a renewed optimism about the ability of students to conceptualize arithmetic operations.

Fermi Problems – The Ultimate Test of Number Sense

CORBIS



Enrico Fermi (1901–1954)

In 1938, the Nobel Prize for Physics was awarded to Italian physicist Enrico Fermi for his work on artificial radioactivity induced by neutrons and its use in controlled chain reactions. A year later, Fermi escaped from war-torn Europe to the United States where he applied his discoveries to the development of the atomic bomb. A gifted teacher and brilliant scientist, Fermi is also known to mathematics educators by his legendary "order-of-magnitude" estimations known as *Fermi problems*.

What is a Fermi Problem?

Throughout his career, Fermi had an uncanny ability to mentally estimate quantities for which there seemed to be insufficient information. For example, Fermi once mused, "How many piano tuners are there in Chicago?" As is typical with most Fermi problems, the initial response would seem to be, "I need more information before I can even conjecture an estimate." However, the essence of obtaining an appropriate estimate is to pose a series of subordinate questions whose combined answers will yield an estimate that is the right order of magnitude (i.e., the closest power of 10).

How Do You Solve a Fermi Problem?

The following sequence of questions, answers, and estimates pertaining to the problem presented above is one of many different routes to a plausible estimate.

Question	Answer
What is the population of Chicago?	About 3×10^6
To estimate the number of pianos, should we estimate the number of people or the number of households?	Households rather than individuals tend to own pianos.
About how many households in Chicago?	There may be an average of 4 people per household in Chicago, so the number of households is about $3 \times 10^6 \div 4$.
What proportion of households in Chicago have pianos?	Electric keyboards etc. have probably replaced the piano in many households, so maybe about 1 in 10 households has a piano. That would suggest that there are about $3 \times 10^6 \div 4 \div 10$ or 7.5×10^4 pianos in Chicago.
How many piano tuners are needed to service 75 000 pianos?	Some pianos are never tuned while others are tuned frequently. Suppose this averages out to about one piano tuning per year per piano. Then 75 000 piano tunings are needed. Assuming a piano tuner tunes an average of 3 pianos a day and that the piano tuner works about 200 days per year, the number of piano tuners needed is about 75 000 $\div$ 600 or 125.
How many piano tuners are there in Chicago?	There are approximately 125 piano tuners in Chicago.

How Do You Know Whether Your Answer is Reasonable?

The table on page 16 shows a series of questions that can be used to obtain an order-of-magnitude estimate of the number of piano tuners in Chicago. However, the answer is only as good as the assumptions on which it is based. In a classroom setting, different groups of students will come up with different sets of questions and therefore different estimates. When several approaches cluster around estimates of similar magnitude (i.e., the same power of 10 when written in scientific notation), then we can be more confident in the plausibility of the estimate. For example, if we have access to a Chicago telephone book, we can count the number of listings under *piano tuners* in the yellow pages. If there were more than 100 listings, this would suggest that our estimate might be a little low, since many listings may be commercial operations with several tuners. If there were fewer than 20 listings, we might consider our estimate too high.

In many of his estimates, Fermi specified a number (called an *upper bound*) that he regarded as definitely larger than the true value of the quantity he was estimating and another number (called a *lower bound*) that he regarded as definitely smaller than the true value. Since it is unlikely that the true number of piano tuners in Chicago is more than 10 times the estimated number, 125, he might say that the number of piano tuners in Chicago is definitely less than 1250. Similarly, since it is unlikely that the true number of piano tuners is less than one tenth of the estimated number, he could certainly assert that the true number is greater than 12.5. If N denotes the true number of piano tuners in Chicago, we can write these upper and lower bounds for N in mathematical notation as $1.25 \times 10 < N < 1.25 \times 10^3$. (The scientific notation reinforces the idea that these are rough "order-of-magnitude" estimates. However, since scientific notation is not included in the expectations until Grade 8, it is suggested that students in Grade 7 use standard notation to represent their estimates.)

Why Are Fermi Problems Worth Doing?

Fermi problems generate the kind of involvement and thinking processes that are at the root of quantitative literacy. Because important information is missing, students must ask themselves more questions about what they need to know and what they already know. Then they must construct a path of estimates that leads from the knowledge they have to the knowledge they need to acquire. The focus of this activity is on the process rather than the answer – a process that mirrors the "number sense" we apply in everyday life when we make "ballpark" estimates of our fuel consumption, our bank balances, or the time we'll need to mark a class test.

Students who acquire the number sense described above will be better able to check whether the answer on their calculator display or computer printout is reasonable. When they hear that the Canadian federal debt is $583 billion, they will not confuse this with $583 million or $583 trillion, recognizing that millions, billions, and trillions are in the same ratio as 1, 1000, and 1 000 000. They will realize that the odds of winning a lottery are roughly the same whether or not you buy a ticket. The development of number sense using such techniques with this generation of students will go a long way toward eradicating the problem that is currently referred to as *innumeracy*.

Creating a Library of Fermi Problems

Imagine having a library of excellent Fermi problems with appropriate sets of questions and good order-of-magnitude estimates that you could use to check the validity of student estimates. You could sprinkle the Fermi problems and their discussions throughout the school year, injecting them into the program at various times to continue honing students' estimation skills and fostering the development of number sense. However, good Fermi problems are not easy to develop because they have to be embedded in contexts familiar to the student and must involve processes that are within the student's grasp. Fortunately, mathematics educators have collaborated to compile a library of such problems. One such collection is accessible on the Internet at the web site:

http://forum.swarthmore.edu/workshops/sum96/interdisc/fermicollect.html

The list of Fermi problems given below includes some of the problems from that web site plus others that are scattered throughout journals, books, and Internet sites.

- How many kernels of (unpopped) corn are there in a cubic metre of popped corn?
- How many thirteen-year-olds are there in Canada?
- How many gas stations are there in Ontario?
- How many shots on goal will all the goalies on all the NHL teams stop during a regular season?
- How many revolutions will an average automobile tire make during its lifetime?
- How many kilometres of railroad tracks are there in Canada?
- How many jellybeans will fill a one-litre bottle?
- What is the total mass of all the students in your school?
- What is the total number of minutes that all the grade seven and eight students in Ontario spend on the telephone in a single year?
- How many water balloons would it take to fill the school gymnasium?

Try some of these problems with your students. Develop for each problem a set of subsidiary questions and, where possible, obtain close estimates from references to check how close student answers approach the actual value. For example, the number of thirteen-year-olds in Canada can be found on the Statistics Canada website at http://www.statcan.ca (see their web page on page 15 of the Data Management module). Remember, however, that it is important not to stress the closeness of the approximation as much as the process by which the estimate is obtained. Both you and your students will enjoy the challenge!

impact math
focus on the future

PART III

Number Sense & Numeration in Grade 7

Number Sense & Numeration: Grade 7

Overall Expectations

By the end of Grade 7, students will:

- compare, order, and represent decimals, integers, multiples, factors, and square roots;
- understand and explain operations with fractions using manipulatives;
- demonstrate an understanding of the order of operations with brackets;
- understand and explain that exponents represent repeated multiplication;
- use estimation to justify or assess the reasonableness of calculations;
- solve and explain multi-step problems involving simple fractions, decimals, and percents;
- explain, in writing, the process of problem solving using appropriate mathematical language;
- use a calculator to solve number questions that are beyond the proficiency expectations for operations using pencil and paper.

Specific Expectations

(For convenient reference, the specific expectations are coded. For example, N 7-3 denotes the third Number Sense and Numeration expectation in Grade 7.)

Students will:

Understanding Number

N 7-1 - compare and order decimals (e.g., on a number line);

N 7-2 - compare and order integers (e.g., on a number line);

N 7-3 - generate multiples and factors of given numbers;

N 7-4 - explain numerical information in their own words and respond to numerical information in a variety of media;

N 7-5 - represent perfect squares and their square roots in a variety of ways (e.g., by using blocks, grids);

Computations

N 7-6 - perform three-step problem solving that involves whole numbers and decimals related to real-life experiences, using calculators;

N 7-7 - understand that repeated multiplication can be represented as exponents (e.g., in the context of area and volume);

N 7-8 - justify the choice of method for calculations: estimation, mental computation, concrete materials, pencil and paper, algorithms (rules for calculations), or calculators;

N 7-9 - demonstrate an understanding of operations with fractions using manipulatives;

N 7-10 - add and subtract fractions with simple denominators using concrete materials, drawings, and symbols;

N 7-11 - relate the repeated addition of fractions with simple denominators to the multiplication of a fraction by a whole number $\left(\text{e.g., } \frac{1}{2}+\frac{1}{2}+\frac{1}{2}=3\times\frac{1}{2}\right)$.

N 7-12 - demonstrate an understanding of the order of operations with brackets and apply the order of operations in evaluating expressions that involve whole numbers and decimals;

N 7-13 - represent the addition and subtraction of integers using concrete materials, drawings, and symbols;

N 7-14 - add integers, with and without the use of manipulatives;

Applications

N 7-15 - ask “what if” questions; pose problems involving simple fractions, decimals, and percents; and investigate solutions;

N 7-16 - explain the process used and any conclusions reached in problem solving and investigations;

N 7-17 - reflect on learning experiences and describe their understanding using appropriate mathematical language (e.g., in a math journal);

N 7-18 - solve problems involving fractions and decimals using the appropriate strategies and calculation methods;

N 7-19 - solve problems that involve converting between fractions, decimals, and percents.

Activity 1 – Teacher Edition

So How Tall Is the CN Tower Really?

Expectations Addressed

N 7-4 explain numerical information in their own words and respond to numerical information in a variety of media.

N 7-6 perform three-step problem solving that involves whole numbers and decimals related to real-life experiences, using calculators.

N 7-8 justify the choice of method for calculations: estimation, mental computation, concrete materials, pencil and paper, algorithms (rules for calculations), or calculators.

N 7-15 ask "what if" questions; pose problems involving simple fractions, decimals, and percents; and investigate solutions.

N 7-16 explain the process used and any conclusions reached in problem solving and investigations.

N 7-18 solve problems involving fractions and decimals using the appropriate strategies and calculation methods.

Context

The four activities in this unit, unlike those in the other modules, do not have a common theme. They are designed as a series of four independent lessons that can be used any time throughout the year. However, they are all dedicated to a common purpose – the development of number sense.

Activity 1 involves students in estimating the height of the CN Tower relative to a typical student height. The height of Sky Pod is calculated from a scale diagram of the CN Tower. Using data pertaining to the dimensions, mass, capacity, and elevators of this tallest free-standing structure in the world, students create a bank of word problems for solution by their classmates.

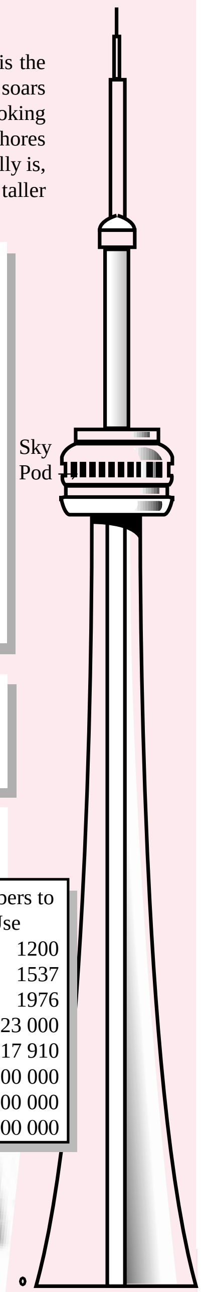
Activity 2 moves from traditional word problems toward *Fermi problems* (see p. 16), by posing the questions, *How long would it take you to count to one billion?* and *How many times will your heart beat in your lifetime?* To address these problems, students must make some general assumptions and then compute some approximations. Students are also prompted to discover the relative sizes of one million and one billion.

In Activity 3, students calculate the height of a stack of loonies required to pay off the Canadian debt. They also solve a variety of single and multi-step word problems involving whole numbers, decimals, and fractions. Then in Activity 4, students tackle the Fermi problem: *Can Spaceship Earth (at EPCOT Center) hold all the world's golf balls?*

Activity 1 – Student Page

So How Tall Is the CN Tower Really?

Most Canadians know that the CN Tower is the world's tallest free-standing structure. It soars over one-half a kilometre into the air, looking down on the City of Toronto and the surrounding shores of Lake Ontario. To understand just how tall this really is, you can calculate how many times the CN Tower is taller than you.



❶ The diagram on the right (drawn to scale) shows a tall person standing beside the CN Tower. Estimate the number of people this height that would be needed to make a tower the height of the CN Tower.

Use a ruler or tape marked in millimetres to estimate how many times the CN Tower is taller than the person in the diagram. Explain how you arrived at your estimate.

The CN Tower is 553.33 m tall. Estimate how many times the CN Tower is taller than you. Measure your height in centimetres. With or without a calculator, determine how many people your height, standing on each other's heads, would stretch upward a distance of 553.33 m.

❷ The world's highest revolving restaurant is located in the Sky Pod (see diagram). Estimate the height of this restaurant above the ground. Show how you obtained your estimate.

❸ Help write this brief information article on the CN Tower by filling in the missing numbers from the list shown here.

The CN Tower– Some Towering Numbers

The CN Tower was officially opened on October 1, ______. Its original cost was ______ dollars, but today it would cost ______ dollars to build. It took a total of ______ construction workers over approximately ______ days to complete the task. Its total mass is ______ tonnes. This is equivalent to the mass of about ______ large elephants. The CN Tower has an annual attendance of ______.

Numbers to Use
1200
1537
1976
23 000
117 910
1 800 000
63 000 000
250 000 000

a very tall person standing beside the CN Tower →

Prior to launching this lesson, tape a long tape measure vertically up a wall so that students can measure their heights in centimetres or millimetres during the activities.

The Lesson Launch

10 minutes

To launch this lesson, you might pose questions such as:

- What are some of the tallest structures in the world?
- What do you think is the *tallest* structure?

Encourage discussion about landmarks such as the pyramid of Cheops (147 m), the Eiffel Tower (320 m), the Empire State Building (443 m), and the Sears Tower (520 m). Invite students who have visited the CN Tower to describe this experience to the class. (The Internet activity on page 25 will help students who have not visited the CN Tower to explore its dimensions visually.) Ask all students to conjecture how many people lying on the ground, head to toe, and forming a human chain it would take to stretch the length of the CN Tower. Record estimates on the blackboard.

Initiating Activity

10 minutes

Group students in pairs and distribute a tape measure marked in millimetres and a copy of page 24 to each pair. Ensure that students are able to identify 1 mm on their tape measures. Assign Exercise ❶ to all pairs. Circulate around the class checking that students are recording lengths of about 250 mm for the height of the CN Tower and about 1 mm for the height of the tall person in the diagram. Estimates in Exercise ❶ should range between 300 and 400.

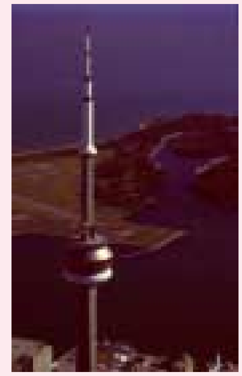
When most students have completed Exercise ❶, ask them how they could determine the distance represented by one millimetre in the picture of the CN Tower on page 24. After it is clear that most students realize that a millimetre in the diagram corresponds to about 2.1 m, invite students to share their estimates in Exercise ❶ with the class. Compare these answers with the original estimates written on the blackboard. Assign Exercise ❷.

When students have finished Exercise ❷, explain that the original cost of the CN Tower was $63 000 000 and today it would cost about four times that. Then assign Exercise ❸.

Activity 1 – Student Page

So How Tall is the CN Tower Really?

You may need your calculator and/or information from Exercise ❸ to help you answer some parts of Exercise ❹. Show your work.


Andrea Pistolesi/The Image Bank

❹ a) A class of 30 Grade 7 students at Pearson Elementary School has a total mass of about 1350 kg. The CN Tower has a mass of about 118 000 t. How many Grade 7 students would have a combined mass the same as the CN Tower?
b) What did it cost per tonne to build the CN Tower?

❺ Here are some interesting facts about the CN Tower. Use these facts to create three word problems. Write the problems in your notebook. Trade word problems with a friend and see if you can solve your friend's word problems. Choose one of you to edit the problems and put them in the class problem bank.

More Interesting Facts About the CN Tower

- There are 6 main elevators to the Look-out level. These elevators travel at a speed of 6 m/s, and take 58 s to reach the Look-out level from the ground. Safety sensors slow the elevators to 1/4 speed during high winds.
- The 6 elevators together can move 1600 passengers per hour.
- The CN Tower has the world's longest staircase, 2579 steps.
- On October 29, 1989, Brendan Keenoy set the world record for the fastest climb of 1760 steps (from ground to Sky Pod, a distance of 342 m) in 7 minutes and 52 seconds.
- Annual attendance is 1.8 million people.
- The 360 Restaurant can seat 400 people and revolves once every 72 min.
- The Space Deck is 447 m above the ground and is the highest observation deck in the world.

Internet

Exploration

To check your answers to Exercises ❷ and ❸, visit the CN Tower web site at:
http://www.cntower.ca

Locate the CN Tower calculator by surfing through the menus or by using the web site address:
http://www.cntower.ca/L1_calc.html

When you enter your height and mass, the calculator will provide you with answers to Exercises ❶ and ❹.

Think-Pair-Share Activity

25 minutes

Distribute copies of page 25 to all students. Have students work independently on Exercise ❹ for 5 minutes. Then group them in pairs to compare and discuss their answers. When most pairs have finished, invite students to report their procedures and results to the class for discussion. Ask why it takes only a few hundred people to stretch as high as the CN Tower, but millions to equal its mass. (Mass, like volume, is proportional to the cube of the linear dimensions.) With students grouped in the same pairs, assign Exercise ❺. Have each student write three word problems on individual slips of paper for the partner to solve. When this task is completed, ask each pair to discuss and edit the word problems to ensure that they make sense and can be solved.

Closure

When all groups have finished, collect the word problems for future use (after your appropriate editing) on class quizzes and tests. Then take up the answers to Exercise ❷. Explain that words like “about” or “approximately” in Exercise ❸ usually denote rounded numbers. Invite the students to revise their answers to Exercise ❸ if necessary before they check them on the Internet. If you cannot get access to the Internet, have students present their answers and explain their reasoning.

SO HOW TALL IS THE CN TOWER REALLY?

Most Canadians know that the CN Tower is the world's tallest free-standing structure. It soars over one-half a kilometre into the air, looking down on the City of Toronto and the surrounding shores of Lake Ontario. To understand just how tall this really is, you can calculate how many times the CN Tower is taller than you.

❶ The diagram on the right (drawn to scale) shows a tall person standing beside the CN Tower. Estimate the number of people this height that would be needed to make a tower the height of the CN Tower.

Use a ruler or tape marked in millimetres to estimate how many times the CN Tower is taller than the person in the diagram. Explain how you arrived at your estimate.

The CN Tower is 553.33 m tall. Estimate how many times the CN Tower is taller than you. Measure your height in centimetres. With or without a calculator, determine how many people your height, standing on each other's heads, would stretch upward a distance of 553.33 m.

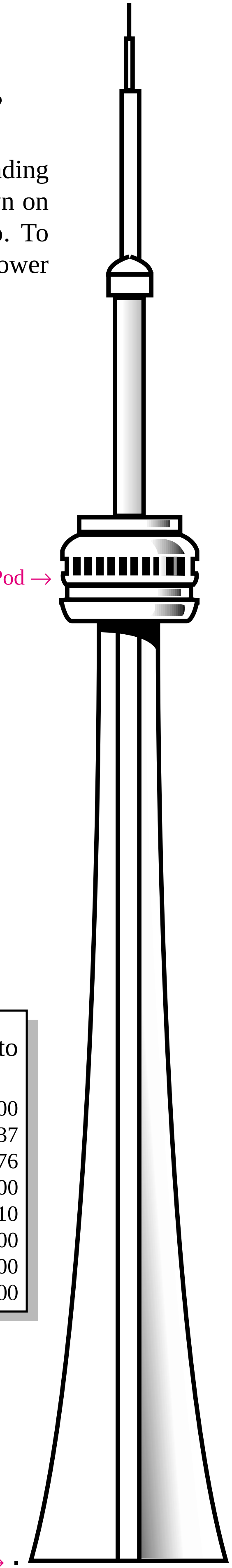
❷ The world's highest revolving restaurant is located in the Sky Pod (see diagram). Estimate the height of this restaurant above the ground. Show how you obtained your estimate.

❸ Help write this brief information article on the CN Tower by filling in the missing numbers from the list shown here.

THE CN TOWER – SOME TOWERING NUMBERS

The CN Tower was officially opened on October 1, ________. Its original cost was ________ dollars, but today it would cost ________ dollars to build. It took a total of ________ construction workers over approximately ________ days to complete the task. Its total mass is ________ tonnes. This is equivalent to the mass of about ________ large elephants. The CN Tower has an annual attendance of about ________ people.

Numbers to Use
1200
1537
1976
23 000
117 910
1 800 000
63 000 000
250 000 000



Sky Pod →

a very tall person standing beside the CN Tower →

You may need your calculator and/or information from Exercise ❸ to help you answer some parts of Exercise ❹. Show your work.

Andrea Pistolesi / The Image Bank

❹ a) A class of 30 Grade 7 students at Pearson Elementary School has a total mass of about 1350 kg. The CN Tower has a mass of about 118 000 t. How many Grade 7 students would have a combined mass the same as the CN Tower?

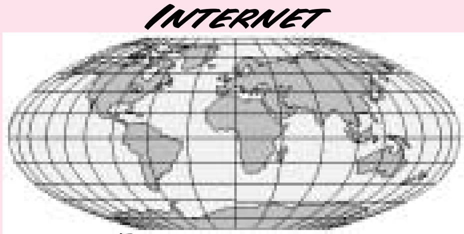
b) What did it cost per tonne to build the CN Tower?

❺ Here are some interesting facts about the CN Tower. Use these facts to create three word problems. Write the problems in your notebook. Trade word problems with a friend and see if you can solve your friend's word problems. Choose one of you to edit the problems and put them in the class problem bank.

MORE INTERESTING FACTS ABOUT THE CN TOWER

- There are 6 main elevators to the Look-out level. These elevators travel at a speed of 6 m/s, and take 58 s to reach the Look-out level from the ground. Safety sensors slow the elevators to 1/4 speed during high winds.
- The 6 elevators together can move 1600 passengers per hour.
- The CN Tower has the world's longest staircase, 2579 steps.
- On October 29, 1989, Brendan Keenoy set the world record for the fastest climb of 1760 steps (from ground to Sky Pod, a distance of 342 m) in 7 minutes and 52 seconds.
- Annual attendance is 1.8 million people.
- The 360 Restaurant can seat 400 people and revolves once every 72 min.
- The Space Deck is 447 m above the ground and is the highest observation deck in the world.

INTERNET



EXPLORATION

To check your answers to Exercises ❷ and ❸, visit the CN Tower web site at:

http://www.cntower.ca

Locate the CN Tower calculator by surfing through the menus or by using the web site address:

http://www.cntower.ca/L1_calc.html

When you enter your height and mass, the calculator will provide you with answers to Exercises ❶ and ❹.

Grade 7

Answer Key for Activity 1

❶ Estimates will vary. However, if the student measures the heights of the tall person and the CN Tower in the diagram, they should obtain lengths of about 1 mm and 252 mm. This would yield an estimate of about 250 people. Actually a person of height 2 m is a very tall person and the CN Tower is 553 m, so it would take about 276 people of that height to form a tower as high. Reasonable estimates may vary between 200 and 500 depending upon whether the people forming the tower are assumed to stand on each others' heads or shoulders. At this point the process of organized thinking to reach an estimate is more important than the actual number obtained.

Most students at this age are somewhere between 1.4 m and 1.8 m in height, so you might expect answers somewhere between 300 and 400 people.

❷ In the diagram, Sky Pod is about 160 mm above the ground and the CN Tower is about 252 mm, so Sky Pod is 160/252 of the height of the CN Tower. Therefore Sky Pod is about 0.63×553 or about 350 m above the ground. (The actual height is 351 m, so the estimate is more accurate than we have a right to expect.)

❸ The completed article is given below.

The CN Tower – Some Towering Numbers

The CN Tower was officially opened on October 1, 1976. Its original cost was 63 000 000 dollars, but today it would cost 250 000 000 dollars to build. It took a total of 1537 construction workers over approximately 1200 days to complete the task. Its total mass is 120 000 tonnes. This is equivalent to the mass of about 23 000 large elephants. The CN Tower has an annual attendance of 1 800 000 people.

❹ a) The mean mass is $1350 \div 30 = 45$ kg. The mass of the CN Tower is about 118 000 t or about 118 000 000 kg. Therefore the number of students would be about

$$118\,000\,000 \div 45 \approx 2\,620\,000.$$

b) It cost $63 000 000 to build the CN Tower and its mass is 117 910 t. The cost per ton was: $63 000 000 ÷ 117910 or about $534 per tonne.

Teacher Note:

The problems that students generate in Exercise ❺ will often show insight and creativity. After they have been submitted and edited by students, check them over to ensure that they are appropriate and store them in a class problem bank with the creator's name opposite the problem. Recycle these problems on unit tests so that students have the opportunity to see their own problems as part of the evaluation. This engenders in the students a greater feeling of control over their program and a deeper sense of involvement.

When taking up the answers to the unit test, discuss some of the student-generated problems. Give credit to the students whose problems have been used. For example, "This is Jamie's problem. Did you find it tricky?" It is particularly gratifying for a low achieving student to have his or her problem used on a test and discussed by classmates.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 1

PROBLEM SOLVING	Level 1	Level 2	Level 3	Level 4
Making Order of Magnitude Approximations (exercises ❶ – ❸) N 7-4, 7-6, 7-16, 7-18	Rarely: • Selects the correct operation to solve a word problem. • Executes the operation correctly using an appropriate tool. • Rounds the approximation appropriately. • Checks to ensure that the approximation is reasonable.	Frequently: • Selects the correct operation to solve a word problem. • Executes the operation correctly using an appropriate tool. • Rounds the approximation appropriately. • Checks to ensure that the approximation is reasonable.	Usually: • Selects the correct operation to solve a word problem. • Executes the operation correctly using an appropriate tool. • Rounds the approximation appropriately. • Checks to ensure that the approximation is reasonable.	Almost Always: • Selects the correct operation to solve a word problem. • Executes the operation correctly using an appropriate tool. • Rounds the approximation appropriately. • Checks to ensure that the approximation is reasonable.

Achievement Levels Defined by the Ministry of Education and Training

Level 1 Identifies achievement that falls much below the provincial standard.

Level 2 Identifies achievement that approaches the standard.

Level 3 The "provincial standard," identifies a high level of achievement of provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.

Level 4 Identifies achievement that surpasses the standard.

WHAT YOU MIGHT SEE

Problem Solving: Use of Computation to Make Order of Magnitude Approximations

Level 1

Activity 1

1 Tallest person 6.8
CN tower 553.33

6.8 x 553.33 = 3762.644

I eastimate 3762.644 people will be needed.

My height 5.4
CN tower 553.33.

5.4 x 553.33 = 2987.982

It would take 2987.982 of me to reach the CN tower

2) Sky pod 1036.66
CN tower 553.33.

553.33 - 1036.66 = 483.33m

The restoront would be 483.33 m above ground.

This student decided that a tall person is 6' 8", and then multiplied the height of the CN Tower, 553.33 m, by 6.8 to determine how many people this height would equal the height of the CN Tower. The selection of the incorrect operation and the confusion of metric and Imperial units suggest that the student has a limited number sense. Similar mistakes are repeated in the other exercises on page 24. All estimates are expressed to several decimal places, suggesting that the student does not understand the significance of order of magnitude approximations.

Level 4

Activity 1 99/01/22

1. My rough estimate is 230 people to form a tower the height of CN Tower.

My estimate with the ruler is 250 times. How I got the answer was easy. I checked if the "very tall person" was a millimeter. Then I checked the CN Tower's height. It was 25 centimeters but since 1 cm is 10 mm I multiplyed 25 by 10 and arrived at my answer 250 times.

My height is 157 cm. The CN Tower is around 352 times taller.

2. Estimated height of this restaurant above the ground is 354m. How I got the answer was simple. I measured the height of the Sky Pod on the diagram and multiplyed it by the answer of 55333cm ÷ 25cm. And got the answer 35413.12 cm. I changed it to meters then rounded down.

3. The CN Tower was officially opened on October 1, 1976. It's original cost was 63 million (63 x 10⁶) dollars, but today it would cost 250,000,000 dollars (250 x 10⁶) to build. It took a total of 1537 construction workers over approximately 1200 days to complete the task. It's total mass is 120000 tons (1.2 x 10⁵). This is equivalent to the mass of about 23000 (2.3 x 10⁴) large elephants. The CN Tower has an annual attendance of 1.8 million people (1.8 x 10⁶).

The student responses presented here show that this student has a well-developed number sense. She began with a rough estimate of 230 tall people to form a tower as high as the CN Tower. Then she performed the appropriate measurements, applied the correct operation and obtained 250. Recognizing that this was close to her original answer, she accepted 250 as reasonable.

In calculating the height of the CN Tower as a multiple of her height, she divided 553.33 by 1.57 to obtain 352.43 and then rounded down to obtain the estimate 352.

In Exercise ❷, the student calculated the scale factor correctly and applied it to her measurement of the height of Sky Pod (16 cm) to obtain the height 35413.12 cm, that she converted to metres and rounded appropriately to 354 m. (The actual height of Sky Pod is 351 m, so her approximation is even closer than we would normally expect from measurements of a diagram with such a large scale factor.)

In Exercise ❸, the student shows exemplary sophistication in her ability to determine which magnitudes are most realistic for the costs, number of construction workers, and annual attendance. If we could raise most students to this level of number sense, we would be close to the eradication of the innumeracy that pervades our society.

WHAT YOU MIGHT SEE

Problem Posing: Posing Problems Involving Whole Numbers & Decimals

Exercise ❺ on page 25 presents the facts shown here and invites students to use these facts to create word problems for other students. A wide variety of word problems were gathered in the field tests. Some were improperly posed while others were very cleverly crafted. Most were two-step and three-step problems. Here is a sample of what we received.

More Interesting Facts About the CN Tower

- There are 6 main elevators to the Look Out level. These elevators travel at a speed of 6 m/s, and take 58 s to reach the Look Out level from the ground. Safety sensors slow the elevators to 1/4 speed during high winds.
- The 6 elevators together can move 1600 passengers per hour.
- The CN Tower has 2579 steps and this is the world's longest staircase.
- On October 29, 1989, Brendan Keenoy set the world record for the fastest climb of 1760 steps (from ground to Sky Pod, a distance of 342 m) in 7 minutes and 52 seconds.
- Annual attendance 1.8 million people
- The 360 Restaurant serves 400 people and revolves once every 72 min.
- The Space Deck, is 447 m above the ground and is the highest observation deck in the world.

① If each elevator went to the Look out level 120 times a day, how many times would all of them together visited the look out level?

Whoops!

② If the 360 restaurant reloves once every 72 minutes, how many times would it rotate in 24 h...

③ If it's annual attendence is 1,8 million since 1976, how many people would have visited it from 1976 to 2007

⑤ 1. Brendan Keenoy set the world record for the fastest climb of 1760 steps up the CN Tower, on October 29,1989. It took him 7 min. 52 sec. Assuming he went the same speed all the way up the stairs, how many steps did he climb per second.

2. There are 6 main elevators that go to the lookout level of the CN Tower. These elevators take 58s to reach the lookout, and back. How many times can one elevator go up and down in one hour (count up and down as one).

3. You can use your answer to question 2 to help you solve question 3.
All of the 6 elevators can take 1600 people per hour. How many people can one elevator carry in one trip?

⑤ a) The 6 elevator at the CN tower together can move 1600 passengers per hour. How many people would 1 elevator move per hour assuming all the elevator carried the same amount of people per hour.

b) The CN Tower has 2579 steps. How long would it take you to climb the stairs if you took one step per two seconds.

c) If it took up 7min 52 sec to climb the CN Towers 2579 steps. How many steps would you climb per minute.

⑤ 1. How long would it take for the elevator to make 4 trips up to the Sky Pod in a Storm? Hmmm…

2. What is the difference in time between the record trip up to the Sky Pod + The elevator trip

3. If 1/3 of the annual attendance go to the restaurant, how many people don't visit the restaurant in the annual attendance.

5. If 6 elevators can carry 1600 people per hour how many people would it be able to carry for an week if the CN Tower was open 10 h per day?
b. If the 360 Restaurant serves 400 people for 72 min. How many people could they serve in 300 mins.
c) If the space deck is 447m from the ground up how many people would be able to match that height if the person was 150cm tall.

5.
1. If the CN Tower has 2579 steps all together including the Sky Pod, and the Sky Pod has 1760 steps from the ground, and each step is approx. 8 inches, than how high is it from the Sky Pod to the top?

2. There are 6 main elevators to the Look Out Level. They travel at a speed of 6 m/s. The Look Out Level is approx. 342 meters, than how long would it take to reach the Look Out Level from the ground?

3. If the Sky Pod is a distance of 342 m, and the Space Deck is a distance of 447 m above the ground, than how many meters are in between the two sites?

Activity 2 – Teacher Edition

How Long Would it Take You to Count to One Billion?

Expectations Addressed

N 7-2 compare and order integers (e.g., on a number line).

N 7-4 explain numerical information in their own words and respond to numerical information in a variety of media.

N 7-6 perform three-step problem solving that involves whole numbers and decimals related to real-life experiences, using calculators.

N 7-16 explain the process used and any conclusions reached in problem solving and investigations.

Context

This activity is dedicated to helping students understand the relative sizes of a million and a billion. As Paulos (see p. 96) has noted, many people regard millions and billions as huge numbers that are beyond normal comprehension and consequently make statements such as, “The Canadian debt is 583 million dollars…or is it 583 billion dollars? I can’t remember, but it’s huge.” To help students understand the relative magnitudes of millions and billions, it helps to use analogies such as:

> A million hours ago was late in the 19th century, but a billion hours ago was over 100 000 years ago – a pre-historic time when the human population was small and sparse.
>
> If a marble were magnified so that its diameter were one million times as large, it would be a sphere with a diameter of about 13 km. If the marble were magnified so that its diameter were one billion times as large, it would be a sphere the size of the Earth.

To estimate how long it would take to count to one billion, students need to estimate how long it would take to say each number. The easiest way is to assume one number per second and adjust the answer later by simply multiplying by the appropriate factor. Page 32 asks the students to use their calculators to convert one billion seconds to minutes, hours, days, and ultimately to 31.7 years. (A review of the relationships among the units of time is recommended in the *Launch* of this lesson, described on p. 31.) Then they can adjust their estimates according to the number of seconds they assume it would take to count each number. Without further guidance, page 33 is given to the students who will work on Exercises ❶ and ❷ to take them step-by-step through the conversion process.

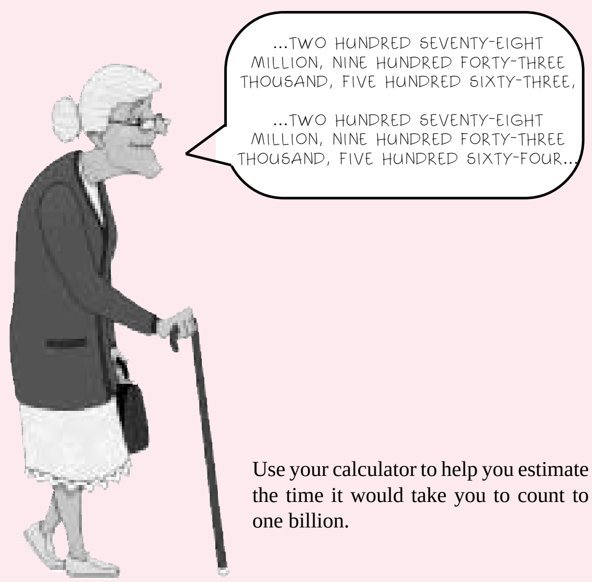
Exercises ❸ and ❹ have students attempt to represent a million and a billion on the same number line so that they realize how miniscule a million is compared to a billion. Finally, in Exercise ❺, students estimate the number of times their hearts will beat in their lifetimes and compare the longevities of their hearts with the longevity of an artificial heart.

Activity 2 – Student Page

How Long Would it Take You to Count to One Billion?

> *"… Many educated people have little grasp for [large] numbers and are even unaware that a million is 1 000 000; a billion is 1 000 000 000; and a trillion, 1 000 000 000 000."*
>
> *Innumeracy*
>
> John Allen Paulos

Suppose you counted to one billion, starting at one and announcing each number in order until you reached one billion. Imagine that you were able to do this without stopping to eat, drink, or sleep. Estimate how many years it would take you to reach one billion. Describe what strategies you used to make your estimate.



Use your calculator to help you estimate the time it would take you to count to one billion.

The Lesson Launch 5 minutes

To launch the lesson, ask students questions such as, “What is larger, a million or a billion?” and “How many millions does it take to make a billion?” Then review some fundamental relationships between time units by asking:

- How many seconds in a minute?
- How many minutes in an hour?
- About how many hours in a day?
- About how many days in a year?

Group students in pairs so that no pair has two weak students. Then ensure that each pair has at least one calculator and distribute page 32. Ask students to use their calculators to estimate how long it would take them to count to one billion.

Paired Activity 15 minutes

Circulate around the class to assist pairs of students who are experiencing difficulty obtaining estimates. Distribute page 33 to the students who need help and suggest that they work through Exercises ❶ and ❷. Distribute page 33 to the other pairs of students after they have completed their estimates. Have them work through Exercise ❷ to check the estimate they made when working on page 32.

Individual Activity 10 minutes

When students have obtained reasonable estimates of the time it would take to count to one billion (anything from 31.7 years to about 10 times that much), ask them to work individually on Exercises ❸, ❹, and ❺ and to record their work in their notebooks. The intent here is to ensure individual involvement by having each student attempt to locate one million on a number line up to one billion and, in so doing, develop a sense of the relative magnitudes of millions and billions. Also, each student will have an individual assumption regarding their own longevity and will obtain a unique estimate of the number of times their heart will beat in their lifetime. As students draw their number lines, circulate around the class and discuss with them why it is difficult to locate one million on this number line.

Activity 2 – Student Page

How Long Would it Take You to Count to One Billion?

You may use your calculator to help you answer these exercises.

❶ Draw a diagram or display calculations to show how you would calculate the number of seconds in an hour; in a year. Display your answers.

❷ Assuming you could count one number every second,write your answers to the following questions in the spaces provided. Show your work on the side.

The time it would take to count from 1 to 1 billion would be approximately:

a) _____________ seconds.
b) _____________ minutes.
c) _____________ hours.
d) _____________ days.
e) _____________ years.

Do you think you could count one number every second? Explain why or why not. If not, how long do you think it would take? Use this new estimate to estimate the approximate number of years it would take to count to one billion if you did not need to stop for food or sleep.

❸ a) What is one million as a fraction of one billion?
b) What is one million as a percent of one billion?

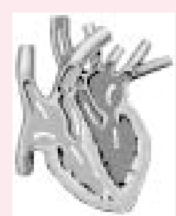
❹ Numbers from 1 to 1 billion can be represented on a number line like the one shown here. Draw a number line like this and place the numbers 250 000 000 and 500 000 000 on it.

Show the locations of these numbers on your number line.

a) 100 000 000 b) 10 000 000 c) 1 000 000

Describe any difficulty you have locating any of the numbers on your number line.

❺ Eventually, an artificial heart may be developed that is capable of beating up to one billion times. Estimate how many times your own heart will beat in your lifetime.



Make reasonable assumptions about:

HINT
- the average number of times your heart beats in a minute.
- the number of years you expect to live.

Do you think your heart has a longer lifetime than such an artificial heart? Explain why or why not.

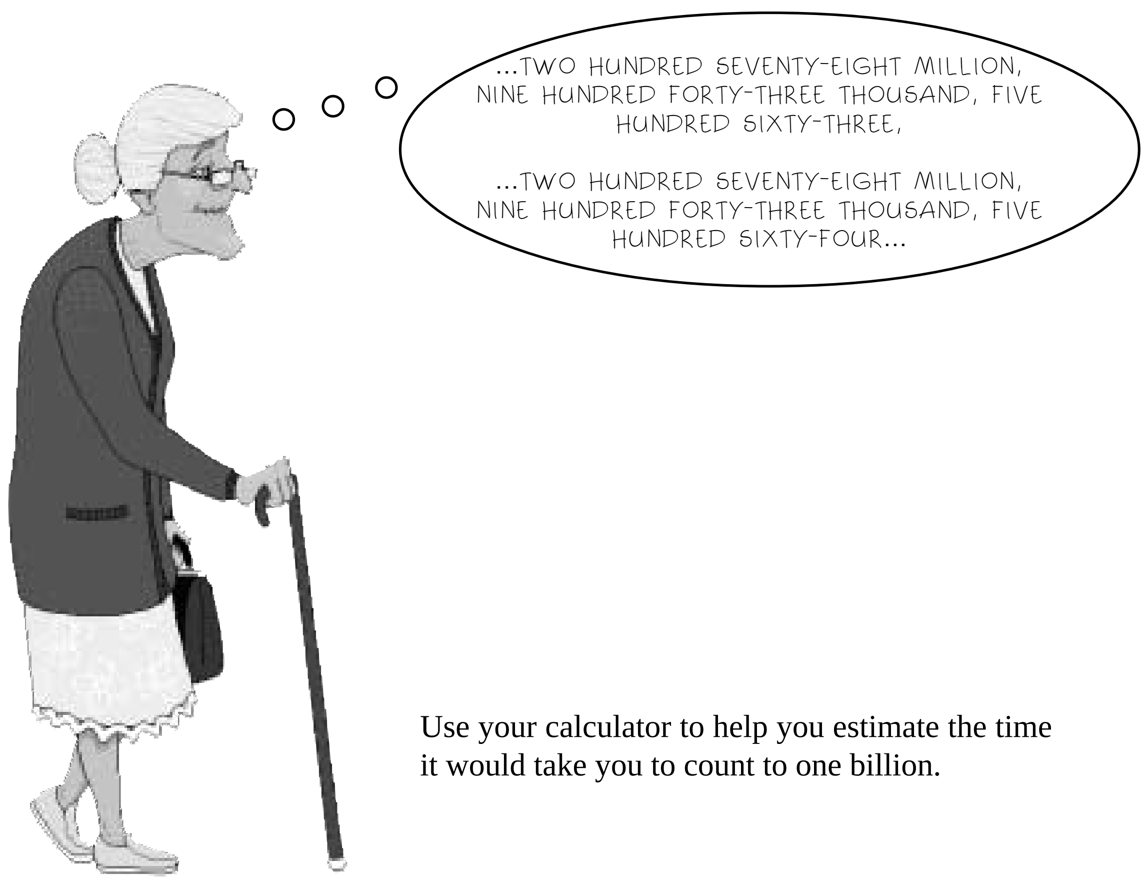
Closure

For many or all of your students, this may be their first exposure to Fermi problems. If so, some will be uncomfortable with the rough approximations involved in assuming an average number of heartbeats per second, in assuming a particular lifespan, or in assuming that every year has 365 days. In their calculator computations, they will tend to carry all the decimal digits shown on the display and may regard rounded answers as inaccurate. It is important at this point to discuss the problem on page 32 and explain that most calculations in real-world problems involve such estimates and that such problems require “order-of-magnitude” answers. That is, we want to know roughly how an artificial heart with a lifetime of one billion beats compares with a human heart. Is its longevity ten times as great as the human heart or only a fraction? Such questions can only be answered by the Fermi-type assumptions and approximations used in this activity. Invite students to suggest other Fermi problems. For each problem, discuss the information that would be needed and the kinds of assumptions that may be required.

How Long Would it Take You to Count to One Billion?

> *"… Many educated people have little grasp for [large] numbers and are even unaware that a million is 1 000 000; a billion is 1 000 000 000; and a trillion, 1 000 000 000 000."*
>
> *Innumeracy*
> John Allen Paulos

Suppose you counted to one billion, starting at one and announcing each number in order until you reached one billion. Imagine that you were able to do this without stopping to eat, drink, or sleep. Estimate how many years it would take you to reach one billion. Describe what strategies you used to make your estimate.



Use your calculator to help you estimate the time it would take you to count to one billion.

Activity 2 – Student Page

You may use your calculator to help you answer these exercises.

❶ Draw a diagram or display calculations to show how you would calculate the number of seconds in an hour; in a year. Display your answers.

❷ Assuming you could count one number every second,write your answers to the following questions in the spaces provided. Show your work on the side.

The time it would take to count from 1 to 1 billion would be approximately:

a) ______________ seconds.

b) ______________ minutes.

c) ______________ hours.

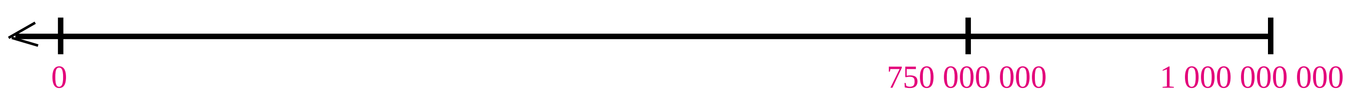
d) ______________ days.

e) ______________ years.

Do you think you could count one number every second? Explain why or why not. If not, how long do you think it would take? Use this new estimate to estimate the approximate number of years it would take to count to one billion if you did not need to stop for food or sleep.

❸ a) What is one million as a fraction of one billion?
b) What is one million as a percent of one billion?

❹ Numbers from 1 to 1 billion can be represented on a number line like the one shown here. Draw a number line like this and place the numbers 250 000 000 and 500 000 000 on it.

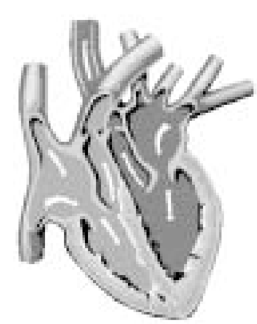


Show the locations of these numbers on your number line.

a) 100 000 000 b) 10 000 000 c) 1 000 000

Describe any difficulty you have locating any of the numbers on your number line.

❺ Eventually, an artificial heart may be developed that is capable of beating up to one billion times. Estimate how many times your own heart will beat in your lifetime.



HINT Make reasonable assumptions about:
- the average number of times your heart beats in a minute.
- the number of years you expect to live.

Do you think your heart has a longer lifetime than such an artificial heart? Explain why or why not.

Grade 7

Answer Key for Activity 2

❶ There are various ways this can be done. Since most children are more comfortable with the concept of multiplication, a natural process for converting years to seconds is the following.

$$\begin{aligned} 1 \text{ year} \times 365 &= 365 \text{ days} \\ &= 365 \times 24 \text{ hours} \\ &= 365 \times 24 \times 60 \text{ minutes} \\ &= 365 \times 24 \times 60 \times 60 \text{ seconds} \\ &= 31\,536\,000 \text{ s} \end{aligned}$$

❷ The time it would take to count from 1 to 1 billion would be approximately:

a) 1 000 000 000 seconds.

b) $1\,000\,000\,000 \div 60 \approx 1\;6\,666\,666.67$ minutes.

c) $16\,666\,666.67 \div 60 \approx 277\,777.78$ hours.

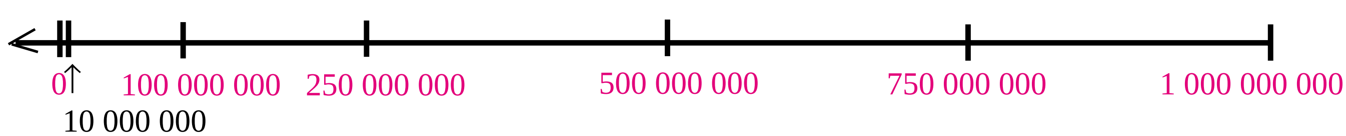
d) $277\,777.78 \div 24 \approx 11\,574.07$ days.

e) $11\,574.07 \div 365 \approx 31.7098$ years.

Most students will realize that they could not count a number every second – especially as the numbers get large. If they estimate 3 seconds to count each number, then they need only multiply the number of years by 3, i.e., $3 \times 31.7098 \approx 95.1293$ years, rather than recalculating the number of years in 3 billion seconds.

❸ a) one million is one thousandth of one billion.
b) one million is 0.1% of one billion.

❹ The number line is shown below with 250 000 000 and 500 000 000 marked on it.



When the students attempt to represent the number 1 000 000 on the number line, they will discover that one million, as large as it sounds, is miniscule compared to a billion, and therefore is too close to 0 to be shown on the number line.

❺ Fermi problems are order-of-magnitude calculations based on a set of assumptions that are taken to be "approximately" true. The assumptions students must make in this problem involve an estimate of personal lifespan and average heart rate throughout life. Often students will assume a lifespan of about 75 years and a heart rate of one beat per second. With these assumptions, the heart will beat about $75 \times 31\,536\,000$ or 2.36×10^9 times. To adjust this estimate to other assumptions such as an average heart rate of 72 beats per minute, merely multiply this estimate by 72/60 or 1.2.

Since any pair of reasonable assumptions about heart rate or lifespan will yield between 2 and 3 billion beats in a lifetime, it is expected that students will conclude that their hearts will have a longer lifespan than such an artificial heart.

The scoring guide presented below has been developed using student responses on a field test conducted in 1999. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 2

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Solution of Multi-Step Problems Involving Estimates** (exercise ❷ & ❺) N 7-6, 7-16	• Estimation procedure is inappropriate. • Calculations are rarely carried out correctly. • Some assumptions are unreasonable.	• Estimation procedure is inappropriate. • Calculations are frequently carried out correctly. • Assumptions are reasonable but may be implicit.	• Estimation procedure is appropriate. • Calculations are usually carried out correctly. • Assumptions are reasonable but may be implicit.	In addition to Level 3: • Calculations are displayed and almost always carried out correctly. • Assumptions are stated explicitly and justified. • The estimate of human heart longevity is used to compare it to the artificial heart.
CONCEPTS **Comparing & Ordering Integers On a Number Line** (exercises ❸ & ❹) N 7-2	Responses satisfy 0 or 1 of the following criteria: • 1 000 000 is expressed as 1/1000 of 1 000 000 000. • 1 000 000 is expressed as 0.1% of 1 000 000 000. • numbers are placed in the correct order on the number line. • each number in the billions is placed relative to 0 in proportion to its size. • there is recognition that 1 000 000 is too close to 0 to be shown on the number line.	Responses satisfy 2 of the following criteria: • 1 000 000 is expressed as 1/1000 of 1 000 000 000. • 1 000 000 is expressed as 0.1% of 1 000 000 000. • numbers are placed in the correct order on the number line. • each number in the billions is placed relative to 0 in proportion to its size. • there is recognition that 1 000 000 is too close to 0 to be shown on the number line.	Responses satisfy 3 of the following criteria: • 1 000 000 is expressed as 1/1000 of 1 000 000 000. • 1 000 000 is expressed as 0.1% of 1 000 000 000. • numbers are placed in the correct order on the number line. • each number in the billions is placed relative to 0 in proportion to its size. • there is recognition that 1 000 000 is too close to 0 to be shown on the number line.	Responses satisfy 4 or 5 of the following criteria: • 1 000 000 is expressed as 1/1000 of 1 000 000 000. • 1 000 000 is expressed as 0.1% of 1 000 000 000. • numbers are placed in the correct order on the number line. • each number in the billions is placed relative to 0 in proportion to its size. • there is recognition that 1 000 000 is too close to 0 to be shown on the number line.

WHAT YOU MIGHT SEE

Understanding of Concepts: Comparing & Ordering Integers on a Number Line

Level 1

3) a) $\frac{1000\,000}{1000\,000\,000}$

B) ?

4) a) 1 000 000 100000000 100000 250 000 000 500 000 000

B) This is much too dificult!

The numbers 100 000 000 and 100 000 are placed in the wrong order on the number line and distances of the numbers from 0 are not proportional to their sizes. There is no evidence of understanding.

Level 2

❸ a) What is one million as a fraction of one billion? $\frac{1\,000\,000}{1\,000\,000\,000}$

b) What is one million as a percent of one billion? 1000000 %

❹ Numbers from 1 to 1 billion can be represented on a number line like the one shown here. Draw a number line like this and place the numbers 250 000 000 and 500 000 000 on it.

0 c) B) 250 000 000 500000000 750 000 000 1 000 000 000 a)

Show the locations of these numbers on your number line.

a) 100 000 000 b) 10 000 000 c) 1 000 000

All the numbers except 100 000 000 are placed in the correct order on the number line. Distances of the numbers from 0 are proportional to their sizes. There is no evidence that the student understands that 1 000 000 is too close to 0 to be distinguished from it on the number line.

Level 3

3 a) $\frac{1}{1000} = \frac{2}{500} = \frac{4}{250} = \frac{8}{125}$

b) 6%

The student has written the correct fraction 1/1000, and then attempted to simplify it by doubling the numerator and halving the denominator. This caused the percentage also to be incorrect.

4 100000000 500000000

0 250000000 750000000 1000000000

I couldn't find where 1,000,000 should go, it's hard because first I have to put 100s of billions, then start using 10s of billions.

All the numbers, are placed in the correct order on the number line. Distances of the numbers from 0 are proportional to their sizes. The student's comment suggests that she understands that 1 000 000 is too close to 0 be distinguished from it on the number line.

Level 4

3 a) $\frac{1,000,000}{1,000,000,000} = 1/1000$ b) $1/1,000 = 0.001 \times 100 = 0.1\%$

4 125,000,000 250,000,000 500,000,000 750,000,000

0 b) 10000000 a) 100000000 c) 1000000 1000,000,000

All the elements identified in the scoring guide on page 35 are here. There is strong evidence that this student understands the relative magnitudes of one million and one billion.

WHAT YOU MIGHT SEE

Problem Solving: Solution of Multi-step Problems Involving Estimates

Level 3

My estimate on counting to billion 159 years. I decided on a number. This number is how long it takes me to say one number. It was 5 seconds. I can count fast at first but it gets slower. I think it would be more than 5 seconds but it is a easy number to work with. I figured out how many numbers I would say in a year. Then I divided it from one billion.

5. My estimation is 294336900 times. I think my heart has longer lifetime than the artificial one because my heart was always with me. The artificial one, you have to take out the other heart to replace it. It's not very healthy.

The student has correctly computed that it would take 158.54 years to count to 1 000 000 000 if it took 5 s per number. The student then correctly rounded this up to 159 years. The assumption of 5 s per number was explained and the process for calculating the estimate was given but detailed calculations were not shown.

Although the student has not stated his assumptions, his estimate corresponds to the assumption that his heart beats about 80 times per minute and that his lifespan is 70 years. The student does not use this estimate to compare the estimated longevity of his heart with that of an artificial heart.

Level 4

I estimate it would take me 63.420 years to count to one billion. I did this by estimating I could count one number in two seconds. I know it would take me 2,000,000,000 s to count to 1 billion. I divided that by 60 to get how many min, that by 60 to get hr, that by 24 to get days, that by 365 to get years.

I do not think I could count 1 number every second. Sure, maybe at the beginning with numbers like 1, but at the end with numbers like 999,999,999, I think I could count 1 number every two seconds. When I'm counting the small numbers, I may take less time, but when I'm counting the big numbers I may take more, so it will even out. The estimate would be: it would take me 63.420 years to count to 1,000,000,000.

← The student has correctly calculated that it would take 63.42 years to count to one billion if it took 2 seconds per number.

← The student has explained the assumptions used in the calculation and given a detailed description of the computations involved.

← The student provides a rationale for the assumption that she could count a number every two seconds.

5. My heart beats about 75 times per minute 4500 per hr. 108000 per day. 39420000 per year and 3,153,600,000 per lifetime (of 80 years). Yes I do, because I am healthy and I excercise, and I already calculated my heart will need to beat more than 3 billion times if I live to 80 (the artificial could only beat 1 billion).

← The student has indicated her assumptions that her heart will beat an average of 75 times per minute and that she will live about 80 years. She has performed the computations correctly and she has also compared this estimate with the one billion beats of an artificial heart and concluded that her heart will probably have the greater longevity.

Activity 3 – Teacher Edition

How High Is the Canadian Debt?

Expectations Addressed

N 7-4 explain numerical information in their own words and respond to numerical information in a variety of media.

N 7-6 perform three-step problem solving that involves whole numbers and decimals related to real-life experiences, using calculators.

N 7-15 ask “what if” questions; pose problems involving simple fractions, decimals, and percents; and investigate solutions.

N 7-16 explain the process used and any conclusions reached in problem solving and investigations.

N 7-18 solve problems involving fractions and decimals using the appropriate strategies and calculation methods.

N 7-19 solve problems that involve converting between fractions, decimals, and percents.

Context

This activity presents two- and three-step word problems involving large numbers. These problems are embedded in an analysis of the Canadian federal debt – its meaning, its magnitude, and its growth. In the lesson launch, students discuss the meaning and origin of government debt. In the initiating activity, they read, interpret, and discuss a graph showing the growth of the federal debt from 1940 to the present and conjecture how high a stack of loonies would be needed to repay the debt. In Exercises ❶ and ❷ of this activity, students test their conjectures by actually performing the calculations.

The height of this stack is then compared to the height of the CN Tower and the distance from the earth to the moon. These calculations provide students with an opportunity to convert from millimetres to metres and to kilometres and, in so doing, conceptualize the relative sizes of millions and billions.

Activity 3 – Student Page

How High is the Canadian Debt?

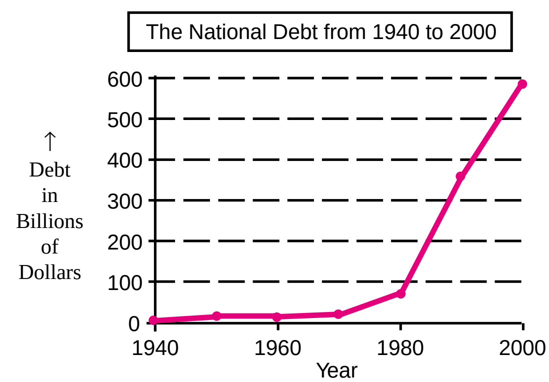
"*a million dollars, a billion, a trillion, whatever. It doesn't matter as long as we do something about the problem.*"

Innumeracy
John Allen Paulos

On February 24, 1998, the Canadian Minister of Finance announced that the national debt will remain at $583.2 billion until the year 2000. That's a lot of loonies!

To understand the magnitude of this debt, try to imagine a vertical stack of 583.2 billion loonies. How high would the stack be? Would it be taller than the CN tower with a height of 553 m? Would it reach as high as an airplane flying at 10 000 m? Or would it reach to the moon – a distance of about 400 000 km?

Pat LaCroix / The Image Bank



Use your calculator to help you estimate the height of a stack of 583.2 billion loonies.

Exercise ❸ presents two-step word problems in which students determine the value of a 50-km chain of loonies placed edge-to-edge, and the length of a chain of one million loonies. The transition from two-step to three-step problems is made in Exercise ❹ in which students combine the concepts of rates, percents, and conversions among time units to solve problems associated with the production of coins. The concept of rate is further extended in Exercise ❺ with the computation of *per capita* debt. Before launching this activity, it is recommended that you review with students the conversion of fractions and decimals to percents. It is also important to review the factors involved in converting among millimetres, metres, and kilometres.

The Lesson Launch 5 minutes

Begin this lesson with a discussion of government debt. "What does it mean to say that the government debt is a billion dollars?" Explain the government's source of revenue (taxes) and the nature of its expenditures (e.g., military, social programs, health care, and foreign aid). Explain also that the annual deficit is the difference between the revenues and expenditures when the government spends more than it collects. Students should understand that the debt is the accumulated value of all the deficits of the past and that it needs to be repaid eventually.

Initiating Activity 10 minutes

Distribute page 40 to the students and have a student read the newsclip. Ask students to estimate how high they think a stack of 583 billion loonies might be relative to the CN Tower. Record their estimates on the blackboard for later comparison.

Ask students to study the graph. Then pose these questions orally to the class.

- About what year did the debt first rise above 100 billion dollars?
- What was the debt in 1990?
- When was the debt about half as much as it is now?
- About how many times greater is the debt now than it was in 1980?

Display a transparency of this graph on the overhead projector and have students come to the overhead and point to the graph on the transparency as they justify their answers.

Individual Activity 30 minutes

Distribute page 41. Assign Exercises ❶, ❷, and ❸ to all students to complete individually. Circulate to help students who are having difficulty. While most students are finishing these exercises, invite students who have correctly completed them to write their detailed solutions, showing all steps, on an overhead transparency. When all students have finished, have the students with completed transparencies display their solutions and explain them step-by-step. Compare the calculated height of the debt-high stack of loonies with the conjectures written on the blackboard and ask students if they think the debt should be repaid in loonies. Then assign Exercises ❹ and ❺ for students to complete in their notebooks. Circulate to help students who have difficulty understanding the questions or with the concepts of rate and percent.

Activity 3 – Student Page

How High is the Canadian Debt?

You may use your calculator to help you complete these exercises.

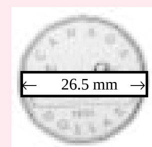
❶ The thickness of a Canadian loonie is 1.95 mm.

a) What is the approximate height of the CN Tower in millimetres?
b) About how many loonies in a stack the height of the CN Tower?
c) What would be the height of a stack of 583.2 billion loonies in millimetres? in metres? in kilometres?
d) Compare your answer in Part c) with your original estimate. Explain how you obtained your estimate and indicate whether it was close to the actual height.

❷ About how many times would a stack of 583.2 billion loonies stretch from the Earth to the moon and back? Show how you obtained your answer.

❸ Organizers of *The Million Dollar Marathon* for disease research stretched 50 km of tape along the route. Contributors were invited to stick loonies on the tape. Use the fact that the diameter of a loonie is 26.50 mm to answer these questions. Use diagrams or words to explain how you obtain your answers.

a) What is the maximum number of loonies that could be stuck on the tape if the loonies were placed edge-to-edge?
b) What fraction of the tape would be covered with loonies by the end of the marathon if the total contributions reached $1 000 000?

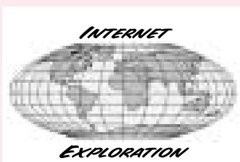


Pat LaCroix / The Image Bank

❹ All Canadian coins are manufactured by the Royal Canadian Mint at its high-speed production plant in Winnipeg. It can produce 100 coins per second. Use this information to answer these questions. Show your work.

a) At the rate of 100 loonies per second, about how long would it take to produce enough loonies to pay off the Canadian debt?
b) What percent of the debt could be paid off by the number of loonies produced if the machines ran 24 hours per day, every day for a year?
c) In 1997, the Royal Canadian Mint produced 16 942 000 twonies. Assuming that all the machines were producing twonies at full capacity, how long would it take to produce these coins?

❺ The *per capita debt* of Canadians is the amount that each Canadian would owe if the entire national debt were shared equally by all Canadians. What is Canada's per capita debt, if the current population of Canada is about 30 million people?



Visit the Royal Canadian Mint web site at:
http://www.rcmint.ca/en/
Find the number of coins of each denomination produced in a recent year. Make up a math problem from this information and invite a friend to solve your problem. Make sure you provide enough information.

Closure

Assign any uncompleted exercises as homework. Those who have completed all the exercises should be encouraged to visit the Royal Canadian Mint web site to collect data and create word problems. These problems should be collected in the next class, edited, and then placed in the class problem bank as described in the "Closure" on page 23.

HOW HIGH IS THE CANADIAN DEBT?

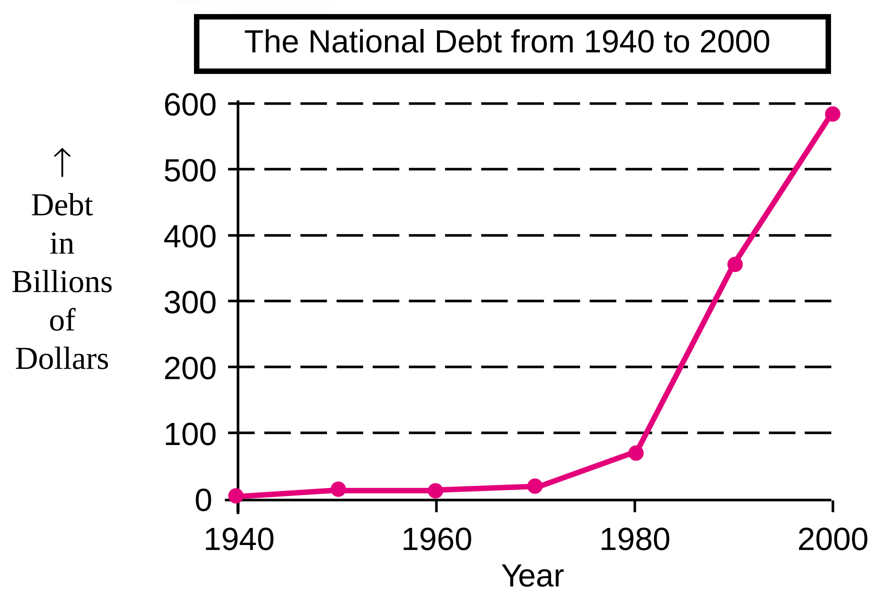
"a million dollars, a billion, a trillion, whatever. It doesn't matter as long as we do something about the problem."

Innumeracy
John Allen Paulos

On February 24, 1998, the Canadian Minister of Finance announced that the national debt will remain at $583.2 billion until the year 2000. That's a lot of loonies!

To understand the magnitude of this debt, try to imagine a vertical stack of 583.2 billion loonies. How high would the stack be? Would it be taller than the CN tower with a height of 553 m? Would it reach as high as an airplane flying at 10 000 m? Or would it reach to the moon – a distance of about 400 000 km?

Pat LaCroix / The Image Bank



Use your calculator to help you estimate the height of a stack of 583.2 billion loonies.

Activity 3 – Student Page

You may use your calculator to help you complete these exercises.

❶ The thickness of a Canadian loonie is 1.95 mm.

a) What is the approximate height of the CN Tower in millimetres?
b) About how many loonies would there be in a stack the height of the CN Tower?
c) What would be the height of a stack of 583.2 billion loonies in millimetres? in metres? in kilometres?
d) Compare your answer in Part c) with your original estimate. Explain how you obtained your estimate and indicate whether it was close to the actual height.

❷ About how many times would a stack of 583.2 billion loonies stretch from the Earth to the moon and back? Show how you obtained your answer.

❸ Organizers of *The Million Dollar Marathon* for disease research stretched 50 km of tape along the route. Contributors were invited to stick loonies on the tape. Use the fact that the diameter of a loonie is 26.50 mm to answer these questions. Use diagrams or words to explain how you obtain your answers.

Pat LaCroix / The Image Bank

a) What is the maximum number of loonies that could be stuck on the tape if the loonies were placed edge-to-edge?

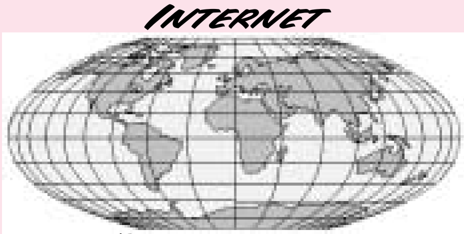
b) What fraction of the tape would be covered with loonies by the end of the marathon if the total contributions reached $1 000 000?

❹ All Canadian coins are manufactured by the Royal Canadian Mint at its high-speed production plant in Winnipeg. It can produce 100 coins per second. Use this information to answer these questions. Show your work.

a) At the rate of 100 loonies per second, about how long would it take to produce enough loonies to pay off the Canadian debt?
b) What percent of the debt could be paid off by the number of loonies produced if the machines ran 24 hours per day, every day for a year?
c) In 1997, the Royal Canadian Mint produced 16 942 000 twonies. Assuming that all the machines were producing twonies at full capacity, how long would it take to produce these coins?

❺ The *per capita debt* of Canadians is the amount that each Canadian would owe if the entire national debt were shared equally by all Canadians. What is Canada's per capita debt, if the current population of Canada is about 30 million people?

Internet



Exploration

Visit the Royal Canadian Mint web site at:
http://www.rcmint.ca/en/

Find the number of coins of each denomination produced in a recent year. Make up a math problem from this information and invite a friend to solve your problem. Make sure you provide enough information.

Grade 7

Answer Key for Activity 3

❶ a) The height of the CN Tower is 553.3 × 1000 mm or 553 300 mm.

b) The number of loonies in a stack the height of the CN Tower would be 553 300 ÷ 1.95 or about 283 744. It would even be appropriate for a student to observe that a loonie is about 2 mm thick and so the number of loonies would be about 553 300 ÷ 2 or about 276 650.

c) A stack of 583.2 billion loonies would have a height of 583 200 000 000 × 1.95 mm or about 1 137 000 000 000 mm. This converts to 1 137 000 000 m and 1 137 000 km.
When students compare this answer with their original estimate, they will probably discover that they underestimated significantly how high a stack of loonies corresponds to the Canadian debt. (This debt should be repaid in paper money of high denomination.)

❷ The distance from the Earth to the moon is about 400 000 km. A stack of 583.2 loonies is about 1 137 000 km. The stack of loones is therefore, about 1 137 000 ÷ 400 000 or 2.8 times the distance from Earth to the moon. This means that it could stretch to the moon and back, and then almost back to the moon again! Practice in working with orders-of-magnitude computations will help students gain an understanding of the relative sizes of millions and billions.

❸ a) The tape is 50 km or 50 000 000 mm long. Each loonie has a diameter of 26.5 mm. Therefore the number of loonies that could be placed on the tape if the loonies are edge-to-edge would be 50 000 000 ÷ 26.5 or about 1 880 000. Answers such as 1.8 or 1.9 million are acceptable.

b) If 1 000 000 loonies are on the tape, then about 1÷ 1.88 ≈ 53/100 or 53% of the tape is covered with loonies. It's acceptable to say that about half of the tape is covered with loonies.

❹ a) As in Activity 2, we can show that the number of seconds in a year is 31 500 000. Therefore the production plant in Winnipeg can produce 3 150 000 000 loonies in a year. The number of years required to produce 583 000 000 000 loonies would be: 583 000 000 000 ÷ 3 150 000 000 or about 185 years!

b) In ❹ a, we found that if the machines ran 24 hours per day every day, the number of loonies equal to the entire debt would be produced in 185 years. Therefore in one year 1/185 or about 0.5% of the number of loonies required to pay off the debt would be produced.

c) Since we anticipate an answer in days, we can compute the number of seconds in a day as 24 × 60 × 60 or about 86 400 seconds. Since the machine can produce 100 coins per second, it can produce 86 400 × 100 or 8 640 000 coins per day. The number of days required to produce 16 942 000 twonies is about:
16 900 000 ÷ 8 640 000 or almost 2 days.

❺ The per capita debt of Canadians is the total debt divided by the total population. That is:
\$583 000 000 000 ÷ 30 000 000 ≈ \$19 440.
Alternatively, we might say that if the Canadian debt were allocated equally to all Canadians, we would each owe about \$20 000!

Teacher Note

In this unit, we have been intentionally cavalier with issues involving significant digits. We often compare quantities having different levels of precision and carry more digits in our answers than is justified. This is done to avoid the complications of introducing the sophisticated concepts of accuracy and precision. Making estimates rather than seeking precise answers allows us this freedom and enables us to postpone discussing issues that students should deal with much later. We have also spelled out the time units, years, days, minutes, and seconds rather than using the formal symbols y, d, min, and s, because these are not yet in universal use and may confuse.

The scoring guide presented below has been developed using student responses on a field test conducted in 1999. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 3

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Solution of Multi-Step Problems Involving Calculations** (exercise ❶ – ❺) N 7-6, 7-16, 7-18, 7-19	• Calculation procedures are inappropriate. • Calculations are rarely carried out correctly. • Answers are rarely checked to determine whether reasonable.	• Calculation procedures are inappropriate. • Calculations are frequently carried out correctly. • Answers are sometimes checked to determine whether reasonable.	• Calculation procedures are usually appropriate. • Calculations are usually carried out correctly. • Calculations are usually displayed and procedures explained.	• Calculation procedures are almost always appropriate. • Calculations are almost always carried out correctly. • Calculations are almost always displayed and procedures explained. • Where possible, answers are checked to ensure they are reasonable.

WHAT YOU MIGHT SEE

Problem Solving: Solution of Multi-step Problems Involving Calculations

Level 3

1. a) The approximate height of the CN Tower in mm is 553330 mm.

b) 553330 ÷ 1.95 = 283758,9 loonies

c) in millimetres – 1137240000000 mm

in metres – 1137240000 m

in kilometres – 1137240 km

d) My estimate – 1166400 km

The Actual height – 1137240 km

To obtain an estimate of the height of 583.2 loonies I did the following operations:

1. First I measured the thickness of a loonie (approximately 2 mm).

2. Then I multipled with the calculator 583.2 x 10^9 (10^9 = a billion) x 2 (the thickness of the loonie) = 1166400000000 mm

3. After that I transformed the result in Km (1166400)

The answers are close.

← An appropriate calculation procedure is used and the calculations are carried out correctly.

← The calculation procedure is explained.

2. About 686 times

400 000 ÷ 583.2 = 685,871

← The calculation procedure is incorrect and yields a meaningless answer.

3. 50 km = 50000000

a) 50000000 ÷ 26.50 = 1886792 loonies

← An appropriate calculation procedure is used and the calculations are carried out correctly.

b) 26.5 km

1000000 x 26.5 mm = 26.5 km

← The calculation procedure is incomplete, giving no answer for the percent of the tape that is covered.

4. a) 583.2 billion ÷ 100 = 5832000000 sec.

5832000000 sec. = 67500 days

← An appropriate calculation procedure is used and the calculations are carried out correctly, but the answer has not been converted to years.

b) 0.5 %.

← The calculation procedure is appropriate and carried out correctly.

First I found out how many seconds are in a year and multipled them with one hundred. Then I transformed them in %.

← The calculation procedure is explained.

c) 16942000 ÷ 100 = 169420 sec. = 1.18 days

← The procedure is appropriate, but there is an error in converting seconds to days.

5. 583.2 Billions ÷ 30 mill = $19440

← The calculation procedure is appropriate and applied correctly.

WHAT YOU MIGHT SEE

Problem Solving: Solution of Multi-step Problems Involving Calculations

Level 4

My estimated height of the loony is 1,1 664,00,000m.

1. a) 553,000 mm.
b) about 283, 590 loonies
c) 1,137,240,000,000 mm, 1,137,240,000 m, 1,137,240 km.

An appropriate calculation procedure is used and the calculations are carried out correctly.

d) I got the estimate by multiplying 583.2 billion by 2(mm). Since the answer was by mm, I divided a 1000 to get m.

My estimate was pretty close to the actual answer.

The calculation procedure is explained.

2. The stack would reach the Moon and come back then go again and almost reach the Moon second time. 2.8431. I got my answer by dividing 400,000 in to 1,137,240.

An appropriate calculation procedure is used and the calculations are carried out correctly.

The calculation procedure is explained.

3. a) 1,886,792 loonies → 50000000 ÷ 26.5 = answer
b) 53/100 → 1 000 000 x 26.5 = 26 500 000

~~26 500 000~~ / ~~50 000 000~~ → ~~265~~ ÷ 5 / 500 ÷ 5 = 53/100

Appropriate calculation procedures are used and the calculations are carried out correctly and displayed.

4. a) 583 200 000 000 ÷ 100 = 5 832 000 000 s.
5832 000 000 ÷ 60 ÷ 60 ÷ 24 = 67500 Days
67500 ÷ 365 = 184. 9315 → 185 Years

Appropriate calculation procedures are used and the calculations are carried out correctly and displayed.

b) 100 x 60 x 60 x 24 x 365 = 3,153,600,000 per year
3153~~600000~~ / 583200~~0000000~~ → 31536 ÷ 48 / 5832000 ÷ 48 = 657 / 121500

657 x 100 ÷ 121500 = 0.5407 %

Appropriate calculation procedures are used and the calculations are carried out correctly and displayed.

← The answer to ❹ c) is missing.

5. The per capita dept would be $19440.
Dept ÷ Canada Population = per capita dept.

↑ The calculation procedure is appropriate, applied correctly, and explained.

Activity 4 – Teacher Edition

Can Spaceship Earth Hold All the World's Golf Balls?

Expectations Addressed

N 7-4 explain numerical information in their own words and respond to numerical information in a variety of media.

N 7-6 perform three-step problem solving that involves whole numbers and decimals related to real-life experiences, using calculators.

N 7-7 understand that repeated multiplication can be represented as exponents (e.g., in the context of area and volume).

N 7-16 explain the process used and any conclusions reached in problem solving and investigations.

N 7-17 reflect on learning experiences and describe their understanding using appropriate mathematical language (e.g., in a math journal).

N 7-18 solve problems involving fractions and decimals using the appropriate strategies and calculation methods.

N 7-19 solve problems that involve converting between fractions, decimals, and percents.

Context

In the previous activities, students have estimated lengths, heights, and times. In this activity, students estimate volumes. Fermi problems involving the estimation of volume have become popular recently as interview questions for executive candidates by corporations that place a high premium on intelligence. An article in the March 1997 issue of *Forbes Magazine* describes some of the interview techniques that are currently used by hiring agencies (see **http://www.forbes.com/forbes/97/0324/5906146a.htm**).

That article contains, under the heading *Guesstimates*, these two Fermi problems that are used to screen applicants.

- *How many barbers are there in Chicago?*
- *How many golf balls does it take to fill the swimming pool used at the Atlanta Olympics?*

The first of these problems is a variation of the original problem posed by Fermi (see p. 16). The second problem is the basis of the problem posed in this activity. Instead of using the swimming pool context, this activity uses Spaceship Earth at EPCOT. Not only is this more interesting to children, but it requires the estimation of the volume of a sphere as a fraction of the volume of the cube that contains it. (It is not recommended that students use the formula for the volume of a sphere.)

The reason that Fermi problems associated with volume are so popular as tests of insight, is that volume relationships are often counterintuitive. The fact that the volume of a 3-D figure is proportional to the cube of its linear dimensions means that doubling its dimensions multiplies its volume by a factor of eight. The problem posed in this activity is *Can Spaceship Earth hold all the world's golf balls?* With the exponential growth in the number of golfers world-wide, it is not surprising that Spaceship Earth cannot hold all the world's golf balls – what is surprising is the fact that it may be the same order of magnitude! **Before launching this activity, it is recommended that you review the multiplication of powers of 10.** Exercise ❶ revisits the Grade 6 expectation, *determine the relationship between linear, square, and cubic units.*

Activity 4 – Student Page

Can Spaceship Earth Hold All the World's Golf Balls?

Spaceship Earth is an architectural landmark situated at the EPCOT Centre in Disney World. With its dimpled spherical surface, it resembles a giant golf ball with a diameter of 55 m. It looks large enough to hold all the golf balls in the world. In fact, one might ask whether this is possible. To help us decide, we must estimate how many golf balls it could hold if the inside space were entirely filled. Then we must estimate how many golf balls are in the world today.

CORBIS

Think about the dimensions of the spherical Spaceship Earth and about the dimensions of a golf ball. Then estimate the number of golf balls that you think could fit inside a sphere of these dimensions.

Assemble a plastic cubic metre (or make one with tape and rolled newspaper) and ten or more centimetre cubes and display them in the classroom as in the diagram on page 49.

The Lesson Launch 10 minutes

Distribute a copy of page 48 to all students. Ask if any students have been to the EPCOT Center. If so, ask them to describe what they remember about the Spaceship Earth pavillion. Explain that EPCOT is an acronym for the Experimental Prototype Community of Tomorrow as envisaged by Walt Disney. Have students observe that Spaceship Earth is a large geodesic sphere (i.e., two geodesic domes or hemispheres put together). If students have worked through the module *Geometry & Spatial Sense*, mention that Cinesphere is a sphere of diameter 38 m and Spaceship Earth a sphere of diameter 55 m. Ask questions such as:

- *How many times as great as the diameter of Cinesphere is the diameter of Spaceship Earth?* (Answer: $55/38 \approx 1.45$ times as great)
- *How many times as great as the surface area of Cinesphere is the surface area of Spaceship Earth?* (Answer: $1.45^2 \approx 2.1$ times as great)
- *How many times as great as the volume of Cinesphere is the volume of Spaceship Earth?* (Answer: $1.45^3 \approx 3$ times as great)

Explain to students that the surface area and volume of 3-D figures are proportional to the square and cube of their linear dimensions. Therefore, even though the diameter of Spaceship Earth is less than 1.5 times the diameter of Cinesphere, its volume is over three times as great. Hold up a golf ball and ask how many of these they think would fit in Spaceship Earth if it were empty on the inside. Record their estimates on the blackboard.

Cooperative Learning Activity 30 minutes

Arrange students into cooperative learning groups of 3 or 4 and assign the roles of chair, recorder, and reporter to each group. Distribute copies of page 49 to all students and assign Exercises ❶, ❷, and ❸. Encourage them to visit the cubic metre and centimetre cube display as they work through Exercise ❶ to check that their answers are reasonable. Circulate among the groups as they work to ensure that they understand what is asked of them and that they are successful in discovering that a cubic metre can contain exactly one million centimetre cubes. This is an important *perceptual anchor* in the estimation of volume (see the module *Measurement* p. 14).

When the students have finished Exercise ❸, have the reporter of each group report the group findings on one of the exercises. Ensure that everyone understands that in Exercise ❶, they proved that $1 \text{ m}^3 = 1\,000\,000 \text{ cm}^3$.

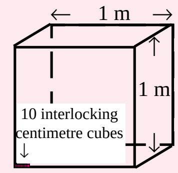
Closure

Assign students the task of completing individually the report described on page 49. Describe Fermi problems and explain that in a Fermi problem the answer is not as important as the assumptions they make and the logical steps they use to break the problem into simpler estimates. For this reason it is very important that they show their work. Direct students who want more information on Spaceship Earth to the Spaceship Earth Fact Sheet at the Internet address: **http://www.intercot.com/edc/Spaceship_Earth/spfacts.html**

ACTIVITY 4 – STUDENT PAGE

CAN SPACESHIP EARTH HOLD ALL THE WORLD'S GOLF BALLS?

❶ Imagine you are placing centimetre cubes in a cubic metre. Answer each question, then fill in the missing number.



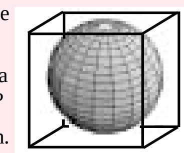
a) An interlocking centimetre cube is a cube with all edges of length 1 cm that can be attached to another centimetre cube. How many interlocking centimetre cubes can be locked together in a *strip* that extends the full length of an edge of a cubic metre?

b) How many of the strips referred to in ❶ a) would it take to form a *layer* of centimetre cubes that cover the bottom of a cubic metre? How many centimetre cubes would it take to form a layer?

c) How many of the layers referred to in ❶ b) stacked on top of each other would fill the inside of a cubic metre?

d) How many centimetre cubes would fill the inside of a cubic metre?

❷ The diagram shows a sphere inscribed in a cube so it touches each face of the cube.



a) Estimate what fraction of the volume of the cube is occupied by the sphere. Explain your thinking.

b) What would be the length, width, and height of a cube just large enough to contain Spaceship Earth?

c) Use ❷ b to estimate the volume of Spaceship Earth.

❸ A golf ball has a diameter of 4.27 cm.

a) Estimate the volume of a golf ball. Explain your thinking.

b) About how many golf balls could Spaceship Earth hold if the golf balls could be packed together with no air space between them?

c) In the tightest packing of spheres, about one quarter of the volume is air space. Use this information to revise your estimate in ❸ b.

REPORT

The total world-wide sales of new and used golf balls is estimated to be between 1.5 and 2 billion dollars annually. Estimate how many golf balls you think there may be in the entire world. Do you think that Spaceship Earth could hold all these golf balls?

Write a report to state your opinion. Support your opinion with:

- your estimates of the number of golf balls that would fit in Spaceship Earth and the number in the world.
- the assumptions you used to obtain these estimates.
- a step-by-step display of your calculations.

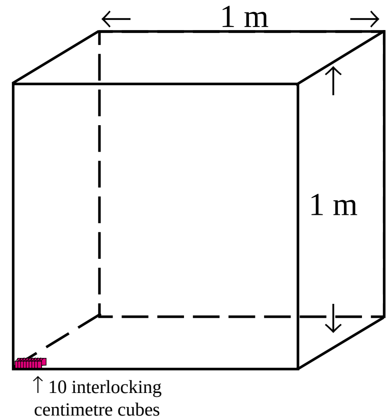
Can Spaceship Earth Hold All the World's Golf Balls?

Spaceship Earth is an architectural landmark situated at the EPCOT Centre in Disney World. With its dimpled spherical surface, it resembles a giant golf ball with a diameter of 55 m. It looks large enough to hold all the golf balls in the world. In fact, one might ask whether this is possible. To help us decide, we must estimate how many golf balls it could hold if the inside space were entirely filled. Then we must estimate how many golf balls are in the world today.

CORBIS

Think about the dimensions of the spherical Spaceship Earth and about the dimensions of a golf ball. Then estimate the number of golf balls that you think could fit inside a sphere of these dimensions.

❶ Imagine you are placing centimetre cubes in a cubic metre. Answer each question, then fill in the missing number.



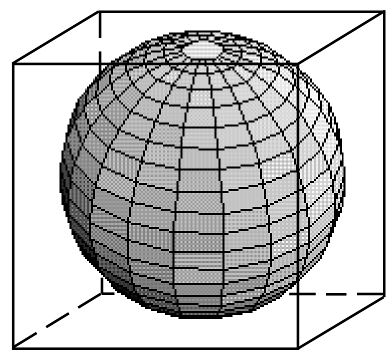
a) An interlocking centimetre cube is a cube with all edges of length 1 cm that can be attached to another centimetre cube. How many interlocking centimetre cubes can be locked together in a *strip* that extends the full length of an edge of a cubic metre?

b) How many of the strips referred to in ❶ a) would it take to form a *layer* of centimetre cubes that cover the bottom of a cubic metre? How many centimetre cubes would it take to form a layer?

c) How many of the layers referred to in ❶ b) stacked on top of each other would fill the inside of a cubic metre?

d) How many centimetre cubes would fill the inside of a cubic metre?

❷ The diagram shows a sphere inscribed in a cube so it touches each face of the cube.



a) Estimate what fraction of the volume of the cube is occupied by the sphere. Explain your thinking.

b) What would be the length, width, and height of a cube just large enough to contain Spaceship Earth?

c) Use ❷ b to estimate the volume of Spaceship Earth.

❸ A golf ball has a diameter of 4.27 cm.

a) Estimate the volume of a golf ball. Explain your thinking.

b) About how many golf balls could Spaceship Earth hold if the golf balls could be packed together with no air spaces between them?

c) In the tightest packing of spheres, about one quarter of the volume is air space. Use this information to revise your estimate in ❸ b.

Report

The total world-wide sales of new and used golf balls is estimated to be between 1.5 and 2 billion dollars annually. Estimate how many golf balls you think there may be in the entire world. Do you think that Spaceship Earth could hold all these golf balls?

Write a report to state your opinion. Support your opinion with:

- your estimates of the number of golf balls that would fit in Spaceship Earth and the number in the world.
- the assumptions you used to obtain these estimates.
- a step-by-step display of your calculations.

Grade 7

Answer Key for Activity 4

❶ a) 100 centimetre cubes.

b) It would take 100 strips each containing 100 centimetre cubes for a total of 10 000 or 10^4 cubes.

c) It would take 100 layers of 10^4 cubes for a total of 10^6 cubes to fill the cube of side length 1 m. This tells us that there are one million cubic centimetres in a cubic metre.

❷ a) On a purely holistic judgement the sphere seems to occupy between half and three-quarters the volume of the cube. However, the formula for the volume of a cube of radius R is $4\pi R^3/3$, so the ratio of the volume of the sphere to the volume of the smallest cube that contains it is actually $\pi/6 \approx 1/2$. Although some of the student samples show the use of this formula, we don't suggest teaching this to students at this point. Holistic estimates between 1/2 or 3/4 are sufficient.

b) Spaceship Earth has a diameter of 55 m. The smallest cube that could contain it would have edges 55 m long. Its volume would be 55^3 m^3 or about 166 375 m^3.

c) The volume of Spaceship Earth is therefore about half of 166 375 m^3 or about 83 000 m^3.

❸ a) The volume of a golf ball is about half the volume of a cube with edges of length 4.27 cm. That is, the volume of a golf ball is about $1/2 \times 4.27^3$ cm^3 or about 39 cm^3.

b) In ❷c), we found that the volume of Spaceship Earth is about 83 000 m^3. Since 1 m^3 = 1 000 000 cm^3, the volume of Spaceship Earth is approximately 83 000 000 000 cm^3. If golf balls could be packed together without spaces between them, then Spaceship Earth could hold approximately 83 000 000 000 ÷ 39, or about 2 100 000 000 golf balls.

c) Since even the tightest packing of spheres leaves about one-quarter of the space unfilled, the space filled by the golf balls is only 3/4 of the total space inside Spaceship Earth. Therefore, Spaceship Earth could hold only 3/4 of the 2 100 000 000 or about 1 575 000 000 golf balls.

Report

The estimation of the number of golf balls in the world is a true Fermi Problem (see p. 16). Several assumptions are necessary and these depend on how the student answers questions such as:

- Does this count all the golf balls at the bottom of lakes and rivers or lost in the woods?
- What proportion of the golf balls sold are new and what proportion are used?
- How many unsold golf balls are on the shelves of retail stores, warehouses, and pro shops?

One possible sequence of questions, answers, and estimates is:

Q. What fraction of all golf balls sold are new?

A. Assume about half the golf balls sold are new.

Q. What is the average sale price of a golf ball?

A. If a new golf ball sells for \$2 and a used golf ball sells for \$1, then the average price for a golf ball is (1/2)(\$2) + (1/2)(\$1) = \$1.50.

Q. About how many golf balls are sold world-wide each year?

A. Assuming total sales of about \$1.8 billion annually and an average sale price of \$1.50, the number of balls sold per year is about:
1 800 000 000 ÷ 1.5 or 1 200 000 000.

Q. About how many *new* balls are sold each year?

A. About half of 1 200 000 000 i.e. 600 000 000.

Q. About how many new golf balls have been produced in the last 40 years?

A. Probably there have been as many golf balls produced in the last 10 years as in the previous 30 years. In the last 10 years there may have been about 6 000 000 000 new balls produced, and therefore about 12 000 000 000 produced in the last 40 years.

Q. Could Spaceship Earth hold all these golf balls?

A. 12 000 000 000 > 1 575 000 000, so Spaceship Earth could probably not hold all the golf balls in the world today.

The scoring guide presented below has been developed using student responses on a field test conducted in 1999. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 4

COMMUNICATION	Level 1	Level 2	Level 3	Level 4
Reporting of Assumptions & Estimates in the Solution of a Fermi Problem **N 7-4, N 7-16, N 7-17, N 7-18.**	Estimates with step-by-step display of calculations of several of the following are incomplete or missing: • the number of golf balls in the world. • the volume of a golf ball. • the volume of Spaceship Earth. Some statements are incoherent and reasoning is unclear.	Report provides estimates with step-by-step display of calculations (in *most* cases) of the following: • the number of golf balls in the world. • the volume of a golf ball. • the volume of Spaceship Earth. Report may or may not state whether Spaceship Earth will hold all the golf balls in the world.	Report provides estimates with step-by-step display of calculations (in *all* cases) of the following: • the number of golf balls in the world. • the volume of a golf ball. • the volume of Spaceship Earth. Report states whether Spaceship Earth will hold all the golf balls in the world and uses the estimates above to support the statement.	• In addition, the report contains clear articulation of the ideas involved and one or more of the assumptions made. • The report uses appropriate units and terminology throughout.

WHAT YOU MIGHT SEE

Reporting of Required Knowledge Related to Concepts, Procedures & Problem Solving

Level 1

I think there would be mabey ten billion golf balls in Spaceship Earth. I think there is probaly eight billion in the hole world.

← Estimates of the number of golf balls that would fit in Spaceship Earth and the number in the world are given.

My assumption is that I gust because there is two billion dollars spent on golf balls, if you put one doller for a used golf ball an two dollers for a brand new golf ball it might be able to come up to my estamate.

← There is an attempt to support the estimate of the number of golf balls in the world by the assumptions that a new golf ball costs two dollars, a used golf ball costs a dollar, and the annual sales of golf balls is between 1.5 and 2 billion dollars. There is no indication how these numbers were used to yield the estimates given.

Frist I tried to make a resnaboll estamato for my ~~result for~~ For my answer I picket ten billion and then I droped it by two billion to get eight billion.

↑ The reasoning presented here is incoherent.

Golfball report

Tare may be about aproxiametly 1 000 000 000 golf balls in the world because 2 billion divided by 1 billion is 1 billion. The 2 billion is money and 1 billion is one golf ball for each dollar which is a billion golf balls.

← An estimate of the number of golf balls in the world is given as one billion, but the reasoning used to obtain this estimate is incoherent.

The amount of golfballs that could fit into spaceship earth is 5.625 million golf balls could fit into spaceship earth because. 1000 000 ÷ 750000 = 7.5 ÷ 4 = 1.875 × 4 = 7.5 and since there still is one quarter of air space 7.5 − 1.875 = 5.625 million.

← An estimate of the number of golf balls that could fit in Spaceship Earth is given as 5.625 million. There is an attempt to display the calculations that yield this estimate, but the reasoning is unclear.

↑ The student has made an effort to adjust the estimate to allow for the fact that one quarter of the volume occupied by the golf balls is air space.

This reports above are generally incoherent. Neither report addresses the original question, *Will all the golf balls in the world fit in Spaceship Earth?* The statements in both reports are generally incoherent and the calculations used to support the estimates, when they are provided, are either incomplete or incoherent.

WHAT YOU MIGHT SEE

Reporting of Required Knowledge Related to Concepts, Procedures & Problem Solving

Level 2

Golf ball problem

golf ball: d = 4.27cm
r = 2.135

$V = 4 \times \pi \times r^3$
$V = 4 \times 3.14 \times 2.135^3$
$V = 4 \times 3.14 \times 9.73^3$
$V = 12.56 \times 9.73^3$
$V = \frac{122.20cm^3}{3}$
$V = 40.73cm^3$

← The student has applied the formula for the volume of a sphere to calculate correctly (to one decimal digit) the volume of a golf ball.

spaceship earth $d = 55m$
$\frac{d}{2} = r$
$r = 27.5$

$V = 4 \times \pi \times r^3$
$V = 4 \times 3.14 \times 27.5^3$
$V = 4 \times 3.14 \times 20,796.87^3$
$V = 12.56 \times 20,796.87^3$
$V = \frac{261208.69}{3}$
$V = 87069.56m^3$

← The student has applied the formula for the volume of a sphere to calculate correctly (to the nearest thousand m^3) the volume of Spaceship Earth..

Since spaceship earth is 8706956cm³ and a golf ball is 40.73cm³ you divide a golf ball into spaceship earth

$\frac{8706956cm^3}{40.73cm^3} = 213772$

This is how many golf balls will go into spaceship earth.

← The division of the volume of Spaceship Earth by the volume of the golf ball is an appropriate procedure, but the conversion from cubic metres to cubic centimetres uses the conversion factor of 100 rather than 1 000 000, causing the estimate of the number of golf balls that would fit in Spaceship Earth to be short a factor of 10 000.

how many golf balls are in the world?

$\frac{2\,000\,000\,000}{2 \times 20}$ because a golf ball costs 2 dollars and the worldwide sale is 2 000 000 000 dollars

← The student has divided by 20 rather than multiplied to determine the total number of golf balls created over a 20-year period.

If you get the answer by doing this you then times the answer by how many years lets say around 20 this would give you an average. Then divide the answer of how many golf balls in the world into how many can fit in spaceship earth if it doesn't fit then spaceship earth can't hold all the worlds golf ball.

← The process described here for determining whether Spaceship Earth could hold all the golf balls is flawed because it suggests dividing, rather than comparing, the two volumes.

It was not intended that students would use the formula for the volume of a sphere in preparing their estimates. However, this student has used it and, generally, applied it correctly and displayed all calculations. Errors discussed above prevented the student from reaching a conclusion on the original question, and this is the main reason for classification of this report as Level 2 rather than Level 3.

WHAT YOU MIGHT SEE

Communication: Reporting of Assumptions and Estimates in the Solution of a Fermi Problem

Level 3

Golf Ball Problem Mon, Feb 1, 1999

QUESTION	ANSWER	
How many golf balls are there in the world?	1.5–2 billion dollars worth sold per year. 1 dollar per ball.	← Assumption stated
When did they start selling golf balls at 1.5–2 billion dollars a year?	I estimate that they started selling golf balls at that rate in 1975. 1999 − 1975 = 24 yrs. 2 billion × 24 = 48 billion	← Assumption stated and estimate given for the number of golf balls in the world. Calculations are displayed.
What is volume of a golf ball?	Volume is $4 \times \pi \times r^3$ $= 4 \times \pi \times 2.136^3 \div 3$ $= 4 \times \pi \times 9.7318103 \div 3$ $= 4 \times 3.14 \times 9.7318103 \div 3$ $= 12.56 \times 9.7318103 \div 3$ $= 122.232 \div 3$ $= 40.744$ cm	← Estimate given for the volume of a golf ball and the calculations are displayed. ← Incorrect units. Should be cm³.
What is the volume of spaceship earth?	Volume is $4 \times \pi \times r^3 \div 3$ $= 4 \times \pi \times 2500\text{cm}^3 \div 3$ $= 4 \times \pi \times 1562500 \div 3$ $= 4 \times 3.14 \times 1562500 \div 3$ $= 12.56 \times 1562500 \div 3$ $= 19625 \div 3$ $= 6541666667$ cm	← Estimate given for the volume of Spaceship Earth and the calculations are displayed. There is an error in calculation of 2500^3. ← Incorrect units. Should be cm³.
What is the volume of all the golf balls in the world?	48 000 000 000 × 40.744 = 1955712000 000 000 No units are given ↑.	← Estimate given for the volume of all the golf balls in the world and the calculations are displayed.

I think that spaceship earth cannot hold all of the golf balls in the world.

← Statement that Spaceship Earth cannot hold all of the golf balls in the world.

This report addresses all the questions. It includes an appropriate strategy for deciding whether Spaceship Earth can hold all the golf balls in the world. Estimates of the number of golf balls in the world, the volume of Spaceship Earth, and the volume of all the golf balls in the world are given and the calculations are displayed. There is an error in computing the volume of Spaceship Earth in converting from linear to volume units. The student uses linear units (cm) rather than volume units (cm³) throughout and in one case gives no units.

WHAT YOU MIGHT SEE

Communication: Reporting of Assumptions and Estimates in the Solution of a Fermi Problem

Level 4

Report - Spaceship Earth

How many golf balls in the world?

- \$1.5 billion of golf balls sold every year.
  ÷ \$2 per golf ball

← Assumption stated

- 750 000 000 golf balls sold every year
  ×25 years golf balls are made
  1.875¹⁰ (they were made more, but are resold so it worksout)

← Assumption stated and estimate given for the number of golf balls in the world. Calculations are displayed.

- There are approximately 1.875¹⁰ (18750000000) golf balls in the world today.

← The student has calculated correctly but used incorrect notation to represent the answer.

How many golf balls will fit into Spaceship Earth?

- $\frac{4\pi r^3}{3}$ is the formula for the volume of a sphere
- 27.5m is the radius (r) for Spaceship Earth
- $\frac{4\pi 27.5^3}{3}$ is the formula for the volume of Spaceship Earth
- 4π = 12.566
- r³ = 20796.875m for Spaceship Earth
- 12.566×20796.875 = 261333.531 m
- $\frac{261333.531m}{3}$ = 87111.177 m³

← Estimate given for the volume of Spaceship Earth and the calculations are displayed. Appropriate units are used.

- The volume of Spaceship Earth is 87111.177m³

- $\frac{4\pi r^3}{3}$ is the formula for the volume of a sphere.
- 2.135 cm is the radius (r) for a golf ball.
- $\frac{4\pi 2.135^3}{3}$ is the formula for the volume of a golf ball
- 4π = 12.566
- r³ = 9.732cm for a golf ball
- 12.566×9.732 = 122.290 cm
- $\frac{122.290cm}{3}$ = 40.763 cm³

The volume of a golf ball is 40.763 cm³.

← Estimate given for the volume of a golf ball and the calculations are displayed.

The volume of Spaceship Earth is in m and that of a golf ball is in cm. So to change 87111.177 m³ (Spaceship Earth's volume) to cm, you move the decimal place over two to the right. This creates 8711117.7 cm³.

← The student recognizes that it is necessary to convert from m³ to cm³ but uses an incorrect conversion factor in converting from m³ to cm³.

- 8711117.7 cm³ volume of Spaceship Earth
  ÷ 40.763 cm³ volume of a golf ball
  213701.585 how many golf balls can fit into Spaceship Earth

← Estimate given for the number of golf balls that can fit into Spaceship Earth and calculations are displayed.

- With an estimate of 1.875¹⁰ golf balls in the world, and Spaceship Earth only being able to hold 213701.585 golf balls, so Spaceship Earth cannot hold all of the golf balls in the world.

← Statement that Spaceship Earth cannot hold all of the golf balls in the world is supported using the estimates presented above.

WHAT DID THE STUDENTS THINK ABOUT THE ACTIVITIES?

Most students seemed to enjoy the activities in the various modules for a variety of different reasons. Here is a sample of their responses to the question, “Did you find this unit interesting? Why or why not?”

yes because it was a chalenge.

Yes I did because the questions are based on an interesting topic that my math book doesn't even have. My favorite was the CN Tower & Billions activityes. Overall this unit was great!

It was really challenging, but yes it was interesting

I found this unit quite interesting because the questions asked in here would be kind of impossible to do.

the unit was interesting. It teaches you many usefull things for life.

I found it interesting because Measurement is one of the units I like.

Yes because it made me find out certain info

I found it a little interesting, some questions were a little boring. I enjoyed the Activity abot Pascal's Law the most.

Yes. The activityes provided me also with interesting information, the broblems are interesting. The language is easy.

Yes, because it has a real life situation involved.

Yes, because it tought me more than math.

yes I found this unit interesting because I got to do graphs with all sorts of numbers and I like to solve questions.

yes, I did find it intesting because I got to learn algerbra and it was easy

Yes, it made me think and go back and try it again,

It was interesting because we are expanding on what we knew.

I found this unit interesting because every time you do a little calculating, you are wondering which place is cheaper.

Yes, because I found the Pascal's triangl very interesting and challenging

Some parts were interesting, comparing the different places, creating the spreadsheet. Making the graphs was kind of frustrating because I had to make sure my calculations were correct.

Yes, I found this unit interesting, because it was a very good challanging piece of work.

I found this unit very interesting because the unit of geometry very challenging. Every question made me used my head. While other books dont always make me do this

Alas, we did not please everyone.

No, because I found it long and somewhat boring

Some of the questions were somewhat hard.

PART IV

Number Sense & Numeration in Grade 8

Number Sense & Numeration: Grade 8

Overall Expectations

By the end of Grade 8, students will:

- compare, order, and represent fractions, decimals, integers, and square roots;
- demonstrate proficiency in operations with fractions;
- understand and apply the order of operations with brackets and exponents in evaluating expressions that involve fractions;
- understand and apply the order of operations with brackets for integers;
- demonstrate an understanding of the rules applied in the multiplication and division of integers;
- use a calculator to solve number questions that are beyond the proficiency expectations for operations using pencil and paper;
- justify the choice of method for calculations: estimation, mental computation, concrete materials, pencil and paper, algorithms (rules for calculations), or calculators;
- solve and explain multi-step problems involving fractions, decimals, integers, percents, and rational numbers;
- use mathematical language to explain the process used and the conclusions reached in problem solving.

Specific Expectations

(For convenient reference, the specific expectations are coded. For example, N 8-3 denotes the third Number Sense and Numeration expectation in Grade 8.)

Students will:

Understanding Number

N 8-1 - represent whole numbers in expanded form using powers and scientific notation (e.g., $347 = 3 \times 10^2 + 4 \times 10 + 7$ and $356 = 3.56 \times 10^2$);

N 8-2 - compare and order fractions, decimals, and integers;

N 8-3 - mentally divide numbers by 0.1, 0.01, and 0.001;

N 8-4 - represent composite numbers as products of prime factors (e.g., $18 = 2 \times 3 \times 3$);

N 8-5 - explain numerical information in their own words and respond to numerical information in a variety of media;

N 8-6 - demonstrate an understanding of operations with fractions.

Computations

N 8-7 - perform multi-step calculations involving whole numbers and decimals in real-life situations, using calculators;

N 8-8 - express repeated multiplication as powers;

N 8-9 - add, subtract, multiply, and divide simple fractions;

N 8-10 - understand the order of operations with brackets and exponents and apply the order of operations in evaluating expressions that involve fractions;

N 8-11 - apply the order of operations (up to three operations) in evaluating expressions that involve fractions;

N 8-12 - discover the rules for the multiplication and division of integers through patterning (e.g., $3 \times [-2]$ can be represented by 3 groups of 2 blue disks);

N 8-13 - add and subtract integers, with and without the use of manipulatives;

N 8-14 - multiply and divide integers;

N 8-15 - understand that the square roots of non-perfect squares are approximations;

N 8-16 - estimate the square roots of whole numbers without a calculator;

N 8-17 - find the approximate values of square roots of whole numbers using a calculator;

N 8-18 - use trial and error to estimate the square root of a non-perfect square;

N 8-19 - use estimation to justify or assess the reasonableness of calculations.

Applications

N 8-20 - demonstrate an understanding of and apply unit rates in problem-solving situations;

N 8-21 - ask “what if” questions; pose problems involving fractions, decimals, integers, percents, and rational numbers; and investigate solutions;

N 8-22 - explain the process used and any conclusions reached in problem solving and investigations;

N 8-23 - reflect on learning experiences and evaluate mathematical issues using appropriate mathematical language (e.g., in a math journal);

N 8-24 - solve problems that involve converting between fractions, decimals, percents, unit rates, and ratios (e.g., that show the conversion of 1/3 to decimal form);

N 8-25 - apply percents in solving problems involving discounts, sales tax, commission, and simple interest.

Activity 1 – Teacher Edition

Mental Math Tricks for Tax & Tips

Expectations Addressed

N 8-2 compare and order fractions, decimals, and integers.

N 8-6 demonstrate an understanding of operations with fractions.

N 8-7 perform multi-step calculations involving whole numbers and decimals in real-life situations, using calculators.

N 8-10 understand the order of operations with brackets and exponents and apply the order of operations in evaluating expressions that involve fractions.

N 8-11 apply the order of operations (up to three operations) in evaluating expressions that involve fractions.

N 8-19 use estimation to justify or assess the reasonableness of calculations.

N 8-20 demonstrate an understanding of and apply unit rates in problem-solving situations.

N 8-22 explain the process used and any conclusions reached in problem solving and investigations.

N 8-24 solve problems that involve converting between fractions, decimals, percents, unit rates, and ratios (e.g., that show the conversion of 1/3 to decimal form).

N 8-25 apply percents in solving problems involving discounts, sales tax, commission, and simple interest.

Activity 1 – Student Page

Mental Math Tricks for Tax & Tips

Many people who serve the public receive a large portion of their income in gratuities (called “tips”). Whether to tip and how much are up to the person being served. However, it is generally recommended that a tip be about 15% of the bill when the service rendered is satisfactory. A particularly pleasant or helpful person may receive a tip closer to 20 or even 25%.

Discuss

❶ a) Why do you think gratuities, rather than wages alone, are paid in the service industries?

b) What are the advantages and disadvantages of the gratuity system to:

(i) the patron? (ii) the server? (iii) the employer?

❷ a) In your group, make a rubric showing what kinds of behaviour would merit tips at the 10%, 15%, and 20% levels for each of the service jobs listed.

	10%	15%	20%
Bellhop			
Server			
Hairdresser			

b) Under what circumstances would you give no tip?

❸ a) Make a table like the one on the right explaining the procedure you would use to calculate a tip at each of the rates, 10%, 15%, and 20%. Look for different ways.

	10%	15%	20%
Procedure for Calculating the Tip	Move the decimal point one place left.		
Expressions for the Tip & the Total Bill	Tip: $0.1x$ Total bill: $1.1x$		

b) If the total bill to which you are adding the tip is x dollars, write an expression for the amount of the tip for each rate. Then write an expression for the total bill including the tip.

Context

The four activities in this unit do not have a common theme. They are designed as a series of four lessons that can be used any time throughout the year in any sequence. However, they are all dedicated to a common purpose – the development of number sense.

Activity 1 presents two familiar contexts in which we are called upon to do mental arithmetic – tipping and taxing. In Ontario, the combined sales tax and GST are 15%. Since a standard gratuity is about 15%, the ability to calculate 15% mentally is a double asset. Students first attempt to develop their own algorithm for calculating 15% mentally. Then they analyse and compare given algorithms and apply them in various contexts.

In Activity 2 students compare fractions to answer the question, *Which culture had the best approximation to π?* The focus on fractions is extended to fraction concepts in Activity 3, where students grapple with two famous fraction paradoxes. Fraction concepts are further expanded in Activity 4 where students investigate what fraction of the numbers in Pascal’s triangle are even. This provides a gentle, intuitive, and pictorial introduction to fractals.

The Lesson Launch
15 minutes

Distribute copies of page 62 to all students. Have students read to themselves the first paragraph. Then discuss Exercise ❶ with the class. Record student responses on the blackboard or overhead, using columns with headings “disadvantages” and “advantages.”

Cooperative Learning Activity
25 minutes

Arrange students into cooperative learning groups of 3 or 4 and assign the roles of chair, recorder, and reporter to each group. Assign Exercises ❷ and ❸ to all groups. Provide each group recorder with an overhead projector pen and two overhead transparencies. Instruct them to use a ruler to make larger versions of the tables on page 62 in which they can record the responses of their group. As you circulate around the classroom, encourage students to move on to Exercise ❸ once they have a couple of entries in each cell of the table in Exercise ❷. Some groups will need help with the algebraic formulation of the tips.

When it appears that all the groups have finished Exercise ❸, have each reporter present the group’s responses on the overhead projector. As this occurs, the class should be checking to ensure that the algorithm presented yields the correct (or approximately correct) tip. Collect and edit the student-generated algorithms at the end of the activity. These can be photocopied and mounted on a classroom poster titled *Mental Math Tricks for Tax and Tips* for future student use.

Individual Activity
20 minutes

Distribute copies of page 63 to all students. Discuss each of the three algorithms presented and compare with the student-generated algorithms. Discuss Exercise ❹ with the class. Assign Exercises ❺ through ❼ to all students to be completed individually in their notebooks.

When the students have finished these exercises, discuss their answers to Exercise ❺. Conduct a mental math quiz by asking questions such as *What is the combined sales tax and GST on \$40? on \$100? on \$72?* Encourage students to answer without using their calculators or pens. Ask students who answer correctly to explain what mental math techniques they used.

Activity 1 – Student Page

Mental Math Tricks for Tax & Tips

The cartoon shows how four different people compute a tip on a bill of \$41.40.

© Taisa Dorney

❹ What is the tip as a percentage of the bill given by each person in the cartoon?

❺ In Ontario, the Provincial Sales Tax (PST) is 8% of the purchase price. The Goods and Services Tax for all Ontarians (GST) is 7%. The combined tax (PST + GST) is 15%. Describe a “mental math way” of calculating the exact combined tax on any amount.
Use your mental math method to estimate the combined tax on these purchases.

a) \$18.00 b) \$24.20 c) \$42.60 d) \$25.99 e) \$35.95

❻ Dana’s hairdresser charges her \$50.00 for a special hair treatment. Then she adds the combined tax. What is the value of the bill:
a) before tax? b) after tax? c) after tax and tip?
d) Do you think Dana should calculate the tip on the \$50 or on the total bill after the tax is added? Give reasons for your answer.

❼ Harvinder sees a \$99.95 jacket on sale at a discount of 20%. The sales clerk adds 15% tax on the normal retail price of \$99.95 and then gives Harvinder a discount of 20% off the total. Harvinder insists that he should only pay tax on the discounted price. The clerk agrees and computes the discounted price first and then adds 15% tax on that amount. Does Harvinder save money by having the discount applied before the tax is computed? Explain why or why not.

Closure

Discuss the answers to Exercises ❻ and ❼. Ask students to explain their reasons why they would calculate the tip before (or after) adding the tax. Invite students to present their solutions to Exercise ❼. Most students will be able to show that Harvinder does not save money by having the price discounted before the tax is applied. In most cases they will show this by calculating the final cost when the tax is calculated before and after the discount is applied and then observing that the final cost is the same in both cases. Display the two computations as \$99.95(1.15)(0.8) and \$99.95(0.8)(1.15) and explain that since a product is not dependent on the order of its factors, both products are the same. That is, it does not matter whether the tax is calculated before or after the discount.

Mental Math Tricks for Tax & Tips

Many people who serve the public receive a large portion of their income in gratuities (called "tips"). Whether to tip and how much are up to the person being served. However, it is generally recommended that a tip be about 15% of the bill when the service rendered is satisfactory. A particularly pleasant or helpful person may receive a tip closer to 20 or even 25%.

Discuss

❶ a) Why do you think gratuities, rather than wages alone, are paid in the service industries?

b) What are the advantages and disadvantages of the gratuity system to:
(i) the patron? (ii) the server? (iii) the employer?

❷ a) In your group, make a rubric showing what kinds of behaviour would merit tips at the 10%, 15%, and 20% levels for each of the service jobs listed.

b) Under what circumstances would you give no tip?

	10%	15%	20%
Bellhop			
Server			
Hairdresser			

❸ a) Make a table like the one on the right explaining the procedure you would use to calculate a tip at each of the rates, 10%, 15%, and 20%. Look for different ways.

b) If the total bill to which you are adding the tip is x dollars, write an expression for the amount of the tip for each rate. Then write an expression for the total bill including the tip.

	10%	15%	20%
Procedure for Calculating the Tip	Move the decimal point one place left.		
Expressions for the Tip & the Total Bill	Tip: $0.1x$ Total bill: $1.1x$		

The cartoon shows how four different people compute a tip on a bill of $41.40.

❹ What is the tip as a percentage of the bill given by each person in the cartoon?

❺ In Ontario, the Provincial Sales Tax (PST) is 8% of the purchase price. The Goods and Services Tax for all Ontarians (GST) is 7%. The combined tax (PST + GST) is 15%. Describe a "mental math way" of calculating the exact combined tax on any amount. Use your mental math method to estimate the combined tax on these purchases.

a) $18.00 b) $24.20 c) $42.60 d) $25.99 e) $35.95

❻ Dana's hairdresser charges her $50.00 for a special hair treatment. Then she adds the combined tax. What is the value of the bill:
a) before tax? b) after tax? c) after tax and tip?
d) Do you think Dana should calculate the tip on the $50 or on the total bill after the tax is added? Give reasons for your answer.

❼ Harvinder sees a $99.95 jacket on sale at a discount of 20%. The sales clerk adds 15% tax on the normal retail price of $99.95 and then gives Harvinder a discount of 20% off the total. Harvinder insists that he should only pay tax on the discounted price. The clerk agrees and computes the discounted price first and then adds 15% tax on that amount. Does Harvinder save money by having the discount applied before the tax is computed? Explain why or why not.

Grade 8

Answer Key for Activity 1

❶ a) Answers will vary, but the main reason would seem to be to create incentives for good service.

b) Although there will be a range of answers, they are likely to include the following:

Advantages:

(i) The patron has the choice to tip as little or as much as he or she chooses based on the level of service provided. This may increase the likelihood of good service.

(ii) The server can be rewarded substantially for offering excellent service.

(iii) The employer can offer a lower wage.

Disadvantages:

(i) The patron sometimes doesn't know how much tip is appropriate or whether the service person expects a tip.
(ii) The server's efforts are sometimes not rewarded when the patron provides an inadequate (or no) gratuity.
(iii) It may be more difficult to attract good workers if the wages are low.

❷ The answers will be more varied and creative than we might imagine in attempting to anticipate student responses.

❸ A number of different algorithms or procedures should be encouraged. Some standard procedures are presented in the table below.

	10%	15%	20%
Procedure for Calculating the Tip	Move the decimal point one place left.	Proceed as for 10% and then increase by half.	Proceed as for 10%, then double the amount.
Expressions for the Tip & the Total Bill	Tip: $0.1x$ Total bill: $1.1x$	Tip: $0.15x$ Total bill: $1.15x$	Tip: $0.2x$ Total bill: $1.2x$

❹ The tips are respectively, 15%, 16.9%, 15%, and 15%.

❺ Any of the methods in Exercise ❹ for calculating 15% is appropriate.
a) \$2.70 b) \$3.63 c) \$6.39 d) \$3.90
e) \$5.39

❻ a) \$50.00 b) \$57.50
c) If the tip is given on the \$50.00, then the total bill including tax and tip is \$65.00. If the tip is given on the \$50 plus tip, then the combined cost is $50(1.15)^2 \approx \$66.13$.
d) Answers will vary because it is an opinion item. It is generally accepted that the tip should be calculated on the original bill before tax because that represents the value of the product and/or service that is provided.

Since a tip of 15% is the same as the combined tax, some people avoid computation altogether by merely leaving a tip equal to the amount of tax displayed on the bill.

❼ If the sales clerk adds 15% to the retail price, and then gives a discount of 20%, the net price to Harvinder is \$99.95(1.15)(0.8).

If the sales clerk gives a discount of 20% first and then adds the 15% tax, the net price to Harvinder is \$99.95(0.8)(1.15).

Multiplication is commutative, i.e., the order of the factors does not change the product, so it does not matter whether we take the discount before or after the tax; the result is the same. Harvinder does not save money by having the discount applied before the tax is computed.

The scoring guide presented below has been developed using student responses on a field test conducted in 1999. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 1

APPLICATION	Level 1	Level 2	Level 3	Level 4
Express the Computation of Sales Tax Algebraically and Apply it Correctly (exercises ❸, ❺, ❻, and ❼) N 8-7, 8-10, 8-25	• Unable to describe verbally or algebraically the procedure for calculating the after-tax cost of an item. • Seldom applies the order of operations correctly in calculating the after-tax cost of an item.	• Able to describe verbally but not algebraically the procedure for calculating the after-tax cost of an item. • Frequently applies the order of operations correctly in calculating the after-tax cost of an item. • Sometimes rounds the after-tax cost appropriately.	• Writes the correct algebraic expression for the computation of the after-tax cost of an item. • Usually applies the order of operations correctly in calculating the after-tax cost of an item. • Usually rounds the after-tax cost appropriately.	• Writes the correct algebraic expression for the computation of the after-tax cost of an item. • Almost always applies the order of operations correctly in calculating the after-tax cost of an item. • Usually rounds the after-tax cost appropriately. • Recognizes that changing the order of the factors does not change the value of a product.

Achievement Levels Defined by the Ministry of Education and Training

Level 1 Identifies achievement that falls much below the provincial standard.

Level 2 Identifies achievement that approaches the standard.

Level 3 The "provincial standard," identifies a high level of achievement of provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.

Level 4 Identifies achievement that surpasses the standard.

WHAT YOU MIGHT SEE

Exercises ❶ and ❷ on page 62 can be used to acquaint students with the process of creating a rubric to assess their own performance on tasks such as completing their homework or maintaining a portfolio. Included here are two samples of student responses to those exercises.

Sample 1

Mental Math Tricks for Tax + Tips 22/01/99

1. a) Well, because you don't really know how hard the person is working, so they rely on their tips.

b) i) the patron =- you might not get good sevice -
-the server relies on your tip-
ii) the server = - your costumer might tip because of sexism, agism.
- your attitude affects the tip.
iii) the employer = - you don't have to pay the employee just based on hours -
- the employees has to work harder for more money.

2. a)

	10%	15%	20%
Bellhop	assitance	smile/attitude assitance	all around helpful assist
Server	service	smile/ service talkative/	Friendly com service nice/good and
Hairdresser	good cut	good cut	attitude/gentle

b) - if the person was rude
- gave little to no service
- got orders wrong

Sample 2

a) I think that gradutties are paid in the service buisness because customers should decide how much the ~~people~~ serving them are paid. If the servant did a very good job, he or she should get a good tip. If, however, a servant did a not so good job at serving the customers, he or she should get a smaller tip.

ii) If it is a server in a popular restaurant, then he will get a lot of customers, wich means a lot of tips. But the downside is, that the servant will get very tired because of the amount of people and serve them worse.

ii) I don't think employers have any advantages in gratuities over the patrons and servants.

2. a)

	10%	15%	20%
Bellhop	opens the door	opens the door and says hello	opens the door, says hello and asks to carry your baggage
Waiter/ Waitress	serves you	pais attention to your table	asks if you need something, and pays attention to the table
Hairdresser	cuts your hair	cuts your hair nicely	freindly, cuts your hair how you wanted it

b) I would not give a tip to the server when he or she is not helpfull at all and behaving like he or she doesn't know that you need some help

WHAT YOU MIGHT SEE

Application: Express the Computation of Sales Tax Algebraically & Apply it Correctly

Level 2

3.

	10%	15%	20%
Procedure for calculated tip	move . one left	divide 10 by half and add that	multiply 10% by 2
Expresson For the Tip	0.1x	$\frac{0.1}{2}$ = ? ? + 0.1	0.1x2

← The student describes correct procedures for calculating tips.

← The student is unable to write algebraic expressions for the tip on an arbitrary amount $ *x*.

4. a) 15% b) 16% c) 15% d) 15%

← The student calculates the percentages correctly, but rounds 16.66% incorrectly to 16%.

5. a) $20.70 b) $27.83 c) $48.99 d) $ 29.88 e) $ 41.34

← The student calculates the after-tax cost rather than the tip, but the calculations are correct.

↑ Small rounding error

6. a) $ 50.00 b) $57.50 c) $66.12

d) I think she should calculate the tip based On the $ 50 bill because the hairdresser should be givin 15% of what her bill is.

← Valid reasoning

7. Yes, because shes pays 20% less than the original, then adds the tax to the 20% less price which totals $91.34

← Student makes a small error in computing the after tax amount. Her answer indicates that she does not realize that changing the order of the factors does not alter the product.

Level 4

3.

a)

	10 %	15 %	20 %
Procedure for calculating the tip	Move the decimal one place to the left.	Multiply x 0.15	Multiply 0.2x
Expression for the tip	Tip: 0.1X Total bill 1.1X.	Tip: 0.15X Total bill: X + 0.15x	Tip: 0.2X Total bill: X + 0.2x

← The student describes correct procedures for calculating tips.

← The student gives correct algebraic expressions for both the tip and the after-tax amount.

4

a) The tip as a percentage of the bill given by each person in the cartoon is 15% Except for 17%

← The student calculates the correct percentages and rounds appropriately, but does not state clearly which of the procedures yields 17%.

5 Slide the decimal point one to the left. Add half this price to itself

a) $ 2.70 b) $3.63 c) $ 6.39 d) $ 3.60 e) $ 5.39

← The student applies the order of operations correctly in computing the tips.

6.

a) $50.00 b) $57.50 c) if 10% tip = $ 63.25

← The student applies the order of operations correctly in computing the tip and the final cost after tax and tip.

7. No, because the clerk does performs the same operation on the price, he just does it in a different order

← This answer indicates that the student realizes that changing the order of the factors does not alter the product.

Activity 2 – Teacher Edition

Is Pi Truly Ubiquitous?

Expectations Addressed

N 8-2 compare and order fractions, decimals, and integers.

N 8-5 explain numerical information in their own words and respond to numerical information in a variety of media.

N 8-6 demonstrate an understanding of operations with fractions.

N 8-9 add, subtract, multiply, and divide simple fractions.

N 8-10 understand the order of operations with brackets and exponents and apply the order of operations in evaluating expressions that involve fractions.

N 8-11 apply the order of operations (up to three operations) in evaluating expressions that involve fractions.

N 8-19 use estimation to justify or assess the reasonableness of calculations.

N 8-22 explain the process used and any conclusions reached in problem solving and investigations.

N 8-23 reflect on learning experiences and evaluate mathematical issues using appropriate mathematical language (e.g., in a math journal).

N 8-24 solve problems that involve converting between fractions, decimals, percents, unit rates, and ratios (e.g., that show the conversion of 1/3 to decimal form).

Context

This activity is dedicated to the investigation of π. Since π plays a pivotal role in the circumference-diameter relationship and since students at this level require some time to digest its significance, this activity is designed to span two class periods. Prior to administering this activity, collect several books that deal with π including the first two sources noted on p. 96, the video mentioned on page 69, and the *Guinness Book of World Records*.

In Exercise ❶ students research some of the rich history associated with the π enigma. In their investigations, students will inevitably confront the idea that there is no fraction that is exactly equal to π. This may be their first encounter with the concept of irrational numbers and (infinite) non-repeating decimal expansions. Fractional approximations to π have been sought by most of the advanced civilizations throughout history. Exercise ❷ involves students in determining which of these fractions is closest to the true value of π. The first four terms of an infinite series that converges to π are given in Exercise ❸, and students must use their calculators to perform several operations with fractions and a square root extraction to convert this to a decimal approximant of π. This exercise involves not only operations with fractions, but also an understanding of the order of operations. These skills are extended in Exercise ❹ where students evaluate a compound fraction to obtain an approximation to π.

Activity 2 – Student Page

Is Pi Truly Ubiquitous?

Math is Cool and Pi is Ubiquitous

It seems that as we approach the 21st century, math is becoming cool. A recent article in a newspaper observed, "In some locales, high school math competitors, *mathletes*, are capable of capturing the limelight once reserved for jocks."

An interest in math is becoming a status symbol and the number *pi* is getting its share of the glory. The movie *Good Will Hunting* cast Matt Damon as a math wizard. The movie *Contact* starred Jodie Foster as an astronomer in search of intelligent life in other galaxies. The award-winning play *Pi* features an investor applying mathematical patterns to reveal stock market trends. Director Darren Aronofsky asserts, "Pi is being rediscovered as a way to look at the universe and find possible answers to eternal questions."

Next year perfume manufacturer Givenchy will launch a new fragrance called Pi. It will be promoted as "the thinking person's scent." If everyone uses this cologne, then mathies will be cool and pi will be truly ubiquitous.

Research the meanings of *pi* and *ubiquitous*.

❶ In your notebook or journal write a short biography of π. Record some of the things you have learned about the mathematical properties of π from your research. Your report should address several different questions including the following:

- What is the approximate numerical value of π?
- Is there a fraction that is equal to π ? If so, what is that fraction?
- Why is π important in mathematics?
- What is a formula in which π appears and how is the formula used?
- Why is π said to be ubiquitous?

A Preparatory Lesson
60 minutes

Distribute copies of page 70 to all students. Read the newsclip to the class and pose the question, *What is π?* Elicit student responses and discuss their meaning. Ensure that all students understand that π (or π:1) is the ratio of the circumference of a circle to its diameter, and that π represents a number that is slightly greater than 3. It is important that students understand that this ratio is the same for all circles no matter what their diameter.

Divide the students into study groups of 3 or 4. Appoint a chair and a reporter for each group. Explain that you will be showing a video, and they are to record in their journals as many answers as they can to the questions on page 70 as they watch the video. Then show a video such as *The Story of π: Project Mathematics!* published by the California Institute of Technology (1988). (If such a video is not available, provide books with information on π and suggest that students perform searches on the Internet to answer the questions on page 70.)

Allow students a full class to gather the information requested in Exercise ❶ to write a biography of π in their journals. For students who have not completed the biography of π by the end of the hour, assign the completion of the biography as homework.

The Lesson Launch
25 minutes

The day following the preparatory lesson, allow students ten minutes to confer with their group reporter. Then call on the reporter in each group to present the group's answers to the questions posed on page 70. Conduct a class discussion of the meaning of *ubiquity*, the significance of π, its numerical value, and a formula or formulas that involve π. Then collect the student journals for assessment of individual work.

Paired Activity
30 minutes

Distribute copies of page 71 to all students. Group the students in pairs and ensure that each pair has a calculator. Assign Exercises ❷ through ❹. Students who finish before the others can should be encouraged to visit the Internet site given on page 71 or to check the *Guinness Book of Records*. In the index under *pi*, they will find information about the record number of digits of π that have been memorized as well as some other entertaining facts about π.

Activity 2 – Student Page

Is Pi Truly Ubiquitous?

π is the ratio of the circumference of a circle to its diameter. Almost every civilization throughout history has attempted to find a fraction equal to the value of π. It was not until 1794 that π was proved to be an *irrational* number. That is, there is no fraction equal to π. However, we can find fractions that are close to π in value and use them as approximations of π. When you press the key for π on your calculator, you obtain a number that has several decimal digits. Is this only an approximation of π?

Approximate Year	Fraction Used to Approximate π	Civilization
2000 B.C.	$3\ ^{1}/_{8}$	Babylonians
2000 B.C.	256/81	Egyptians
250 B.C.	$3\ ^{10}/_{71} < \pi < 3\ ^{1}/_{7}$	Archimedes of Ancient Greece
450 A.D.	355/113	Ch'ung-chih of China
530 A.D.	3927/1250	Aryabhata of India
1220 A.D.	864/275	Fibonacci of Italy

❷ The decimal expression of π correct to 8 decimal places is 3.141 592 65.
a) Which of the civilizations in the table above found the fraction that was the closest approximation to the true value of π?
b) Which of the civilizations had the least accurate approximation of π?
c) Which civilization, India or Ancient Greece, had the closer approximation to π?

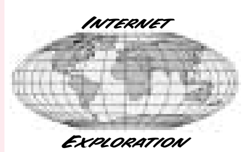
❸ To approximate the value of π, the great mathematician and scientist Isaac Newton used this complicated-looking expression: ➔ $\pi \approx \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{160} - \frac{1}{3584}\right)$

Use your calculator to evaluate the right side of this expression. How many correct digits of π does this expression yield?

❹ A mathematician named Lambert used this expression to approximate π. ➔ $\pi \approx 3 + \frac{1}{\left(7 + \frac{1}{15}\right)}$

Use your calculator to evaluate the right side of this expression. How many correct digits of π does this expression yield? Which expression, the one given here or in Exercise ❸, gives the closer approximation to π?

Watch the order of operations!



To investigate the ubiquity of π, check the web site at: **http://www.joyofpi.com**

Add another paragraph or two to your biography of π. Include the answers to such questions as:
- How many digits of π are known today?
- What is the greatest number of digits of π that have been memorized?

Share with a partner any new information you have discovered.

Closure

When the groups have completed the exercises on page 71, have members of various pairs write on the blackboard the decimal equivalents of the fractions in Exercises ❷ through ❹. Then pose questions such as, *Which fraction is closest to 3.1415926…?* and *Which fraction is the worst approximation to π and why?* If you have access to the book *The Joy of π* by David Blatner (see reference p. 96), you will find a variety of interesting vignettes to read to your class. For a five-minute excerpt, you can read *The Legal Value of Pi*, on pages 104-5, that describes how the Indiana legislature, in 1897, nearly made π equal to 4 *by law*! For a longer session, with a human dimension, you can read excerpts from *The Chudnovsky Brothers* (pp. 65-71).

IS PI TRULY UBIQUITOUS?

Math is Cool and Pi is Ubiquitous

It seems that as we approach the 21st century, math is becoming cool. A recent article in a newspaper observed, "In some locales, high school math competitors, *mathletes*, are capable of capturing the limelight once reserved for jocks."

An interest in math is becoming a status symbol and the number *pi* is getting its share of the glory. The movie *Good Will Hunting* cast Matt Damon as a math wizard. The movie *Contact* starred Jodie Foster as an astronomer in search of intelligent life in other galaxies. The award-winning play *Pi* features an investor applying mathematical patterns to reveal stock market trends. Director Darren Aronofsky asserts, "Pi is being rediscovered as a way to look at the universe and find possible answers to eternal questions."

Next year perfume manufacturer Givenchy will launch a new fragrance called Pi. It will be promoted as "the thinking person's scent." If everyone uses this cologne, then mathies will be cool and pi will be truly ubiquitous.

Research the meanings of *pi* and *ubiquitous*.

❶ In your notebook or journal write a short biography of π. Record some of the things you have learned about the mathematical properties of π from your research. Your report should address several different questions including the following:

- What is the approximate numerical value of π?
- Is there a fraction that is equal to π ? If so, what is that fraction?
- Why is π important in mathematics?
- What is a formula in which π appears and how is the formula used?
- Why is π said to be ubiquitous?

A SHORT HISTORY OF FRACTIONAL APPROXIMATIONS TO PI

π is the ratio of the circumference of a circle to its diameter. Almost every civilization throughout history has attempted to find a fraction equal to the value of π. It was not until 1794 that π was proved to be an *irrational* number. That is, there is no fraction equal to π. However, we can find fractions that are close to π in value and use them as approximations of π. When you press the key for π on your calculator, you obtain a number that has several decimal digits. Is this only an approximation of π?

Approximate Year	Fraction Used to Approximate π	Civilization
2000 B.C.	$3\ ^{1}/8$	Babylonians
2000 B.C.	256/81	Egyptians
250 B.C.	$3\ ^{10}/71 < \pi < 3\ ^{1}/7$	Archimedes of Ancient Greece
450 A.D.	355/113	Ch'ung-chih of China
530 A.D.	3927/1250	Aryabhata of India
1220 A.D.	864/275	Fibonacci of Italy

❷ The decimal expression of π correct to 8 decimal places is 3.141 592 65.
a) Which of the civilizations in the table above found the fraction that was the closest approximation to the true value of π?
b) Which of the civilizations had the least accurate approximation of π?
c) Which civilization, India or Ancient Greece, had the closer approximation to π?

❸ To approximate the value of π, the great mathematician and scientist Isaac Newton used this complicated-looking expression:

$$\pi \approx \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{160} - \frac{1}{3584}\right)$$

Use your calculator to evaluate the right side of this expression. How many correct digits of π does this expression yield?

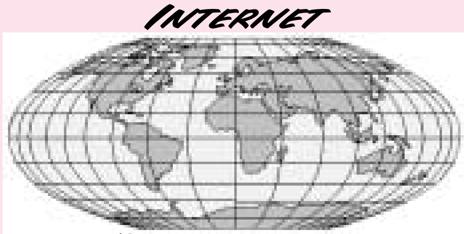
Watch the order of operations!

❹ A mathematician named Lambert used this expression to approximate π.

$$\pi \approx 3 + \frac{1}{\left(7 + \frac{1}{15}\right)}$$

Use your calculator to evaluate the right side of this expression. How many correct digits of π does this expression yield? Which expression, the one given here or in Exercise ❸, gives the closer approximation to π?

INTERNET



EXPLORATION

To investigate the ubiquity of π, check the web site at:
http://www.joyofpi.com

Add another paragraph or two to your biography of π. Include the answers to such questions as:
- How many digits of π are known today?
- What is the greatest number of digits of π that have been memorized?

Share with a partner any new information you have discovered.

Grade 8

Answer Key for Activity 2

❶ At this point students should know that π is close to 3. Some will know that π is about 3.14 and others will give π to whatever number of decimal digits they can obtain on their calculator display.

The student report should indicate that π is irrational, i.e., there is no fraction that is equal to π.

Many different reasons why π is important may be given. Probably the most commonly identified by your students will be the fact that it appears in many formulas and enables us to calculate the circumferences and areas of circles.

The word "ubiquitous" derives from the Latin word *ubique* meaning "everywhere." Since π seems to pop up everywhere in mathematics, we say "π is ubiquitous." Students may discover in their research that π appears in the study of probability (Buffon needle experiment), in statistics (normal distribution), and in almost all branches of mathematics. They may use this fact to verify that π is indeed ubiquitous.

❷ The decimal equivalents of the fractions used to approximate π are shown in the table below.

Fraction Used to Approximate π	Decimal Equivalent
$3\,^1/_8$	3.125
256/81	3.160
$3\,^{10}/_{71} < \pi < 3\,^1/_7$	$3.140 < \pi < 3.143$
355/113	3.14159
3927/1250	3.14160
864/275	3.14181

a) As the table shows, the fraction 355/113 offers the closest approximation to the true value of π.

b) The fraction 256/81 is the least accurate approximation to π.

c) India's fraction 3927/1250 is closer to the true value of π than the fraction 22/7 used by the Ancient Greeks.

❸ The expression used by Newton was actually an infinite series. Only the first few terms of that series are displayed in the given expression on page 71. The decimal expansion of that expression is approximately 3.14234. It is correct to three decimal places and it yields the first three (correct) digits of π. Actually Newton used 22 terms of this infinite series and obtained 17 correct digits of π.

❹ Lambert's expression gives the first 3 terms of the infinite continued fraction for π. Its decimal equivalent is approximately 3.14150. This gives four correct digits of π.

The expression in Exercise ❸ yields 3.14234 and the expression in Exercise ❹ yields 3.14150, so Lambert's expression is slightly closer to the true value of π.

Internet Exploration

To obtain the number of digits to which π has been computed, check the Internet or the most current issue of the *Guinness Book of Records* in the index under π. It will be a number greater than one billion.

The current record for the greatest number of digits of π that have been memorized can also be obtained from the *Guinness Book of Records*. It will be a number greater than 40 000!

The scoring guide presented below has been developed using student responses on a field test conducted in 1999. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 2

	Level 1	Level 2	Level 3	Level 4
APPLICATION **Application of the Order of Operations to Expressions Involving Fractions** (exercises ❶ – ❹) **N 8-2, N 8-9, N 8-10, N 8-11, N 8-24**	• Significant difficulties in comparing fractions. • Major errors and/or omissions in evaluating expressions containing fractions.	• Some errors in comparing fractions. • Several minor errors and/or omissions in evaluating expressions containing fractions.	• Almost no errors in comparing fractions. • A few minor errors and/or omissions in evaluating expressions containing fractions.	• No errors in comparing fractions. • Almost no minor errors and/or omissions in evaluating expressions containing fractions.
COMMUNICATION **Articulation of the Meaning and Properties of Pi** (exercises ❶ – ❹) **N 8-23, N 8-24**	• Responses to the questions posed are seldom complete and clear. • Responses are inconsistent in their use of appropriate mathematical terminology.	• Responses to the questions posed are often complete and clear. • Responses are inconsistent in their use of appropriate mathematical terminology.	• Responses to the questions posed are complete and clear. • Responses include appropriate mathematical terminology.	• In addition to Level 3: the responses contain clear articulation of the ideas involved and are supported by diagrams, formulas, or examples.

WHAT YOU MIGHT SEE

Application of Mathematical Procedures: Evaluation of Expressions Involving Fractions

Level 2

2 a = ch'ung-chih of china is the closest to pi. ← Correct comparison of fractions.

b = Fibonacci of Italy has the least closest. ← Incorrect comparison of fractions.

c = Archmides of Ancient Greece had the closer approximation to π ← Incorrect comparison of fractions.

3. ? ← Evaluation of the expression containing fractions is missing.

4. ? ← Evaluation of the expression containing fractions is missing.

Level 3

(2) a) The closet is Ch'ung-chih of China in 450 A.D. ← Correct comparison of fractions.

b) Egyptians ← Correct comparison of fractions.

c) arckmides of ancient Greece ← Incorrect comparison of fractions.

(3) ? ← Evaluation of the expression containing fractions is missing.

(4) 3.2095 ← In the evaluation of the fractional approximation to π, the student added the reciprocal of 7 and the reciprocal of 15 instead of taking the reciprocal of the sum of 7 and 1/15. This is an error in applying the order of operations.

Level 4

2. The China civilization found the closest fraction ← Correct comparison of fractions.

b The Egyptians were the most off ← Correct comparison of fractions.

C. India was closer ← Correct comparison of fractions.

3 3. 14238095 ← Evaluation of the expression has the first four decimal digits correct.

4. 3.142 : yes. ← The student appears to have evaluated the expression correctly and then rounded up the third decimal digit. It is not clear what was meant by the response "yes."

WHAT YOU MIGHT SEE

Communication: Articulation of the Meaning & Properties of Pi

Level 2

There is a approximate numerical value
of pi it is 3.141592654. But there
is no fraction that can explain π.
π is important to measure circles and other
things. π is used by finding Circum ference
and Diameter in math.

An appropriate numerical approximation to π is given and the student understands that π cannot be expressed as a fraction. A sense of why π is important is evident, but no formula is given. Also the question about the ubiquity of π is not addressed.

Level 3

-1. ubiquitous: being or
seeming to be everywhere
at the sametime

pi: the ratio of the circumference
of any circle to its diameter,
equal to about 3.141592 the
Circumference of a circle pi
times the diameter of the circle
($C=\pi d$).

Questions

- The approximate Value of pi = 3.14
- No there is not a number equal to pi
- pi is important to math because
other wize we coolden't calulate
the circumfrence/area of a circle.
- pi is said to be ubiquitous
because π is everywhere

An appropriate numerical approximation to π is given and the student seems to understand that π cannot be expressed as a decimal number, although the terminology is somewhat awkward. The relationship between the circumference and diameter of a circle includes a formula. The student also explains why π is important. A correct response regarding the ubiquity of π is also given.

Level 4

1. Pi is used in many different eqastions. What is Pi?
Pi, of approximate value of 3.14, is the ratio of the
circumference of any circle to its diameter. It is used
in the equation that shows the area of a circle (πr^2)
This is the international sign used for Pi - π. There
is no actual fraction that is equal to because
in the 1700's it was proved π was an irrational
number there are however, fractions that come
very close to pi for estimation purposes. Why is
π important to math? It's important because π can
quickly and accuratly give you the answers to
problems. They say it is Ubyutious because
it can help in so many problems and unpredictible
places.

An appropriate numerical approximation to π is given and the student understands that π cannot be expressed as a fraction and expresses this idea by referring to the irrationality of π. The formula for the area of a circle in terms of π and its radius is also given. The student also explains why π is important. A correct response regarding the ubiquity of π is also given. This report contains full sentences that respond clearly to the questions asked.

Activity 3 – Teacher Edition

Mathematical Dialogues

Expectations Addressed

N 8-5 explain numerical information in their own words and respond to numerical information in a variety of media.

N 8-9 add, subtract, multiply, and divide simple fractions.

N 8-21 ask "what if" questions; pose problems involving fractions, decimals, integers, percents, and rational numbers; and investigate solutions.

N 8-22 explain the process used and any conclusions reached in problem solving and investigations.

N 8-23 reflect on learning experiences and evaluate mathematical issues using appropriate mathematical language (e.g., in a math journal).

Context

This activity is based on an old Arab tale that is retold in *The Man Who Counted* (see reference p. 96). On pages 15–18 of that book, author Malba Tahan tells of a wealthy Arab, Salem Nasair, who was ambushed and robbed by desert nomads. Two men who happened upon the victim shared their bread with him: one contributed three loaves and the other contributed five loaves. All three men shared equally in the eight loaves. When a reward of eight gold pieces was to be paid to the two contributors there was some discussion regarding how many coins each should receive. Should the coins be divided in the ratio of 3:5 or 1:7? Since the answer is counterintuitive, it is an excellent problem for stimulating a class discussion.

Page 78 presents the same problem, revised so that it has a contemporary context. In this formulation of the problem, Karen and Kevin contribute in the ratio of 9:15 or, in lowest terms, 3:5. The reward to be divided is 24 cans of pop rather than 8 gold pieces, but the appropriate division in both cases is 1:7.

Activity 3 – Student Page

Mathematical Dialogues

© Taisa Dorney

❶ a) What was the total cost of the pizza and soft drinks including tip and tax?
b) How much was contributed by Kevin? by Karen?
c) How many cans of soft drink are to be shared by Karen and Kevin?

❷ Explain how Karen might justify her claim that she is entitled to 9 of the 24 cans of pop.

❸ Explain how Kevin might justify his claim that he is entitled to 21 cans.

❹ How many cans of pop would you give to Kevin and Karen if you had to decide how to share the 24 cans between them? Justify your answer. Use a diagram and/or a logical argument to explain your thinking.

❺ Suppose Kevin had contributed $16 and Karen had contributed $8.
a) How many of the 24 cans of pop would Karen expect to receive?
b) How many would Kevin expect to receive?
c) How many cans would you give to each of them? Justify your answer.

Page 79 presents a modern version of another Arabian problem in which 35 camels are bequeathed to three sons so that the oldest receives 1/2 of them, the second oldest 1/3 and the youngest 1/9. The sons argue because none of these fractions yields a whole number of camels. A passing stranger contributes his camel so that there are now 36 camels. The sons take their shares of 36, namely, 18, 12 and 4 camels, leaving one camel that is reclaimed by the stranger who rides it off into the sunset. In the problem on page 79, students are to receive shares of 1/2, 1/3, and 1/8 of 23 CDs. As in the original Arabian problem, a stranger contributes a CD so that the students can claim 12, 8, and 3 CDs and the stranger retrieves his CD. In both problems, the paradox arises because the fractions do not add up to 1. The stranger merely allocates the missing portion so that all the shares become integers. **Before launching this activity, it is recommended that you review factors, multiples, and common denominators. You may also provide sets of 24 centimetre cubes for students to use as manipulatives.**

Cooperative Learning Activity 1

30 minutes

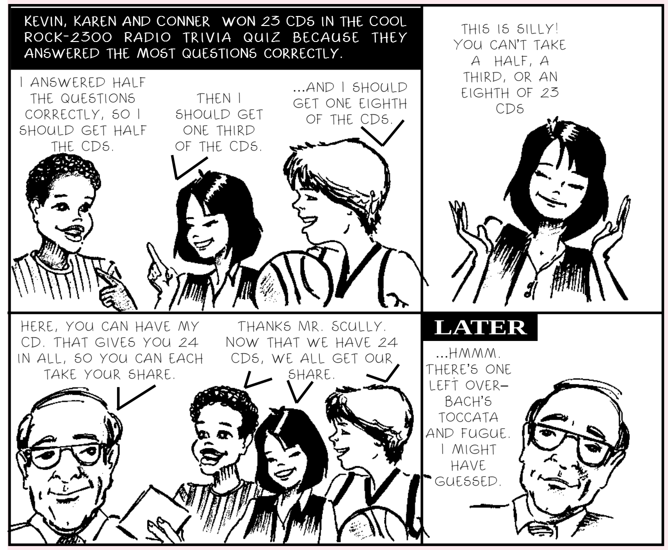
Divide the class into groups of 3 or 4. Appoint a chair, a recorder, and a reporter for each group. Distribute copies of page 78 to all students. Display an overhead transparency of the dialogue on page 78 and read the speech balloons in each rectangle of the dialogue as you display it. Then invite questions to ensure that the students understand the dialogue. Assign each group the task of completing Exercises ❶ through ❺ on page 78. Remind the reporters in the groups that they must be prepared to defend their positions – do they support Kevin's point of view or Karen's?

Circulate around the room as the students discuss the issues. They are expected to assume that Karen, Kevin, and Melissa shared the pizza equally. Usually most students support Karen's claim at first. As more discussion occurs, some students may begin to consider the pizza consumed by each individual and it occurs to them that they must subtract the personal consumptions before they can calculate the respective contributions to Melissa. This brings them to an understanding of Kevin's entitlement. In Exercise ❹, the student is guided toward a "what if?" scenario. That is, what if Karen's consumption is equal to her contribution? Is she entitled to any part of the reward? The "what if" scenario can be carried even further by asking, "What if Karen had contributed less than what she ate?"

When all the groups have completed their reports, invite a reporter to state and justify their group's opinion about how the 24 cans of pop should be distributed. After the opinion is expressed, invite another reporter representing an opposing view to give reasons why the opposing view is more reasonable. Continue to receive reports and encourage discussion until some consensus is reached. If no consensus is reached, read to the class pages 15 through 18 in *The Man Who Counted.* If still no consensus is reached, ask the students to take page 78 home to discuss with their parents and to report on what their parents think.

Activity 3 – Student Page

Mathematical Dialogues



TALK ABOUT IT. © Taisa Dorney

❻ a) How many CDs did Kevin, Karen, and Conner win?
b) What share of the CDs did each of them claim?
c) Why were they unable to receive those shares?
d) What did Mr. Scully, the music teacher, do to resolve the problem?

❼ a) Assuming each of the students received the shares they claimed, how many CDs did each of them receive?
b) How many CDs did the students receive in total?
c) Did Mr. Scully's contribution provide shares of the 23 CDs that were different from the shares claimed by the students? If so, how did the shares change?

❽ Explain why Mr. Scully was able to divide the 23 CDs to everyone's satisfaction even though he did not really change the number of CDs to be divided.

❾ Suppose the students had won 11 CDs and claimed shares of 1/2, 1/4 and 1/6. Can you use Mr. Scully's method to share the CDs to everyone's satisfaction? Explain why or why not.

❿ Create a problem like the one above that can be solved by Mr. Scully's method. Assume that three people are to share some number of identical items and they claim shares of 1/2, 1/3, and 1/9.

Cooperative Learning Activity 2

30 minutes

Divide the class into groups of 3 or 4. Appoint a chair, a recorder, and a reporter for each group. Distribute copies of page 79 to all students. Repeat the procedure followed for the cooperative learning activity above and assign Exercises ❻ through ❿ on page 79 to the groups. When all groups have completed the exercises discuss the answers to Exercises ❻ and ❼. Then invite the reporters to present their answers to Exercise ❽. Students may attempt to explain Mr. Scully's procedure rather than why it works. Guide the students toward the discovery that the fractions 1/2, 1/3 and 1/8 do not add up to 1. Ask students to determine the sum of these fractions. When they have discovered that the sum is 23/24, explain that the teacher merely distributed the unallocated share of 1/24 among each of the students so that they each received a whole number of CDs. (Mr. Scully's method works only when the sum of the shares are a fraction of the form $n/(n + 1)$ for some whole number n.)

Mathematical Dialogues

❶ a) What was the total cost of the pizza and soft drinks including tip and tax?
b) How much was contributed by Kevin? by Karen?
c) How many cans of soft drink are to be shared by Karen and Kevin?

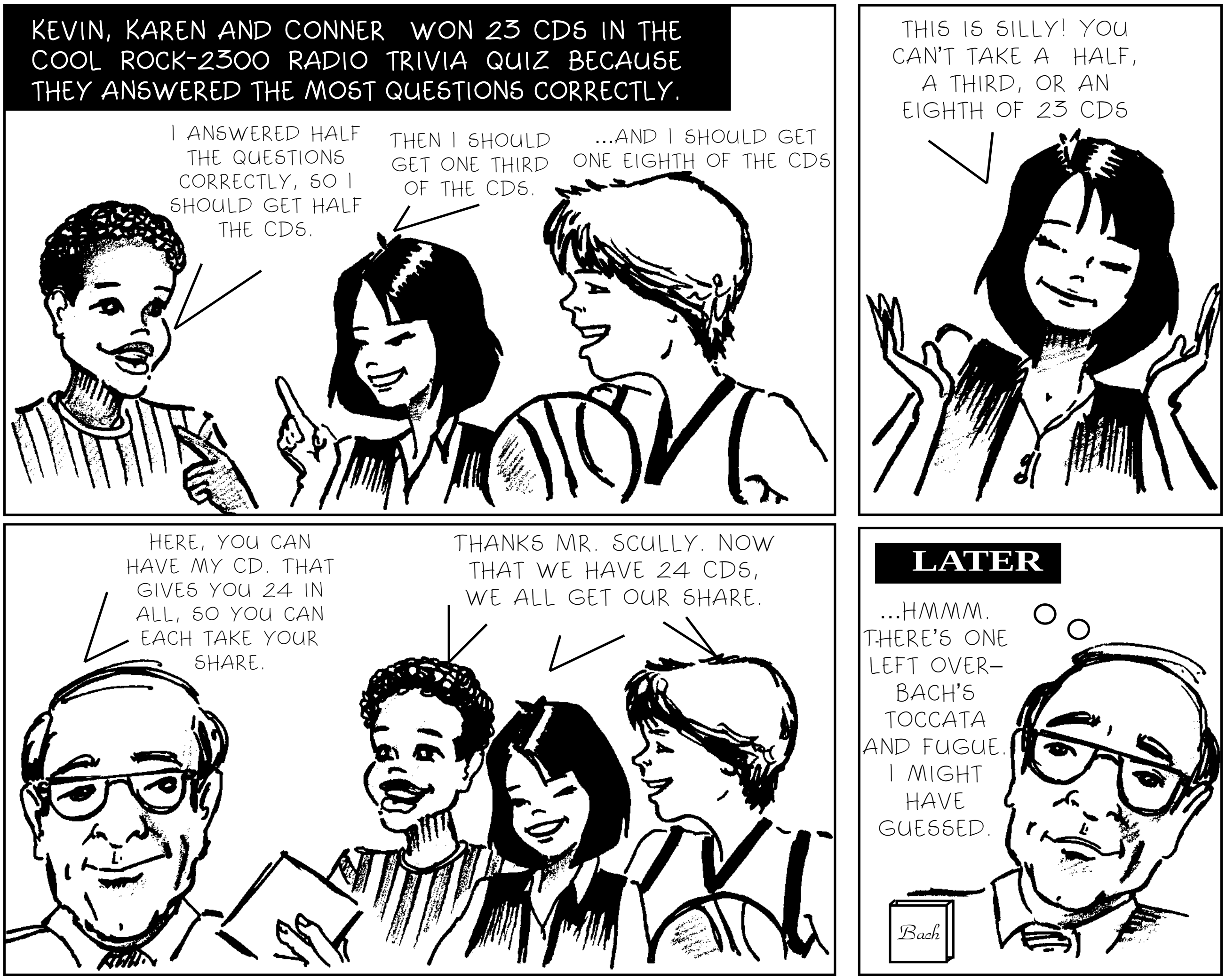
❷ Explain how Karen might justify her claim that she is entitled to 9 of the 24 cans of pop.

❸ Explain how Kevin might justify his claim that he is entitled to 21 cans.

❹ How many cans of pop would you give to Kevin and Karen if you had to decide how to share the 24 cans between them? Justify your answer. Use a diagram and/or a logical argument to explain your thinking.

❺ Suppose Kevin had contributed $16 and Karen had contributed $8.
a) How many of the 24 cans of pop would Karen expect to receive?
b) How many would Kevin expect to receive?
c) How many cans would you give to each of them? Justify your answer.

Mathematical Dialogues



TALK ABOUT IT.

❻ a) How many CDs did Kevin, Karen, and Conner win?
b) What share of the CDs did each of them claim?
c) Why were they unable to receive those shares?
d) What did the music teacher, Mr. Scully, do to resolve the problem?

❼ a) Assuming each of the students received the shares they claimed, how many CDs did each of them receive?
b) How many CDs did the students receive in total?
c) Did Mr. Scully's contribution provide shares of the 23 CDs that were different from the shares claimed by the students? If so, how did the shares change?

❽ Explain why Mr. Scully was able to divide the 23 CDs to everyone's satisfaction even though he did not really change the number of CDs to be divided.

❾ Suppose the students had won 11 CDs and claimed shares of 1/2, 1/4 and 1/6. Can you use Mr. Scully's method to share the CDs to everyone's satisfaction? Explain why or why not.

❿ Create a problem like the one above that can be solved by Mr. Scully's method. Assume that three people are to share some number of identical items and they claim shares of 1/2, 1/3, and 1/9.

Grade 8

Answer Key for Activity 3

❶ a) The total cost of the pizza and soft drinks including tax and tip was $24.
b) Kevin contributed $15 and Karen $9.
c) 24 cans are to be shared by Karen and Kevin.

❷ Karen might observe that she contributed 9/24 of the purchase and therefore she should receive 9/24 of the 24 cans, i.e., 9 cans.

❸ Kevin might observe that each person was to contribute $8. Since Melissa had no money, Kevin contributed $7 toward Melissa's share (i.e., $15 less his share of $8) and Karen contributed $1 toward Melissa's share. Therefore, Melissa owed Kevin $7 and Karen $1. If she resolved the debt by paying in soft drinks instead of dollars, Kevin should receive 7/8 of the 24 cans or 21 cans, and Karen should receive 3 cans.

❹ Answers will vary. Kevin's argument may be regarded as the "mathematically correct" solution. However, the reasoning and justification is more significant than the answer given. It is more important that each student understand the logic of both arguments. Some students might share Karen's point of view out of a sense of "fairness." Social norms generally lean toward equitable distribution of rewards and profits and militate against what may appear to be skewed distribution of gains.

❺ As in Exercise ❹, answers will vary. The logic of Kevin's argument may be a little more evident in this case because Karen's contribution of $8 pays only for her share and Kevin has lent Melissa the entire $8 to cover her share. Therefore, if she chooses to repay her debt in cans, Kevin should receive all 24 cans.
a) 0 cans
b) 24 cans
c) At this point, more students should recognize the logic of Kevin's argument. To those who remain commited to Karen's point of view, ask how the cans should be distributed if Kevin were to contribute $17 and Karen $7, so that in effect Kevin is owed $1 by Karen and $8 by Melissa.

❻ a) 23 CDs
b) They claimed 1/2, 1/3, and 1/8 of the CDs.
c) They could not distribute these shares because 23 is not divisible by 2, 3, or 8.
d) Mr. Scully contributed a CD of his own.

❼ a) When the 24 CDs were divided into shares of 1/2, 1/3, and 1/8, the students received 12, 8, and 3 CDs respectively.
b) The students received 23 CDs in total.
c) The shares of the 23 CDs that the students received were respectively 12/23, 8/23, and 3/23. That is, each student received a slightly larger share than they had claimed. The student who claimed 1/2 actually received a share of 12/23. The student who claimed 1/3 actually received 8/23, and the student who claimed 1/8 actually received 3/23.

❽ The original fractions claimed did not add up to 1, so there was a share of 1/24 that had not been allocated. That is,

$$\frac{12}{24}+\frac{8}{24}+\frac{3}{24}=\frac{23}{24}$$

In mathematical terms, the teacher took the unallocated share of 1/24 and by increasing each of the shares claimed by a factor of 24/23, he assigned the larger shares:

$$\frac{24}{23}\left(\frac{1}{2}\right),\ \frac{24}{23}\left(\frac{1}{3}\right),\ \text{and}\ \frac{24}{23}\left(\frac{1}{8}\right)$$

That is, he increased the shares to 12/23, 8/23, and 3/23. These fractions total one, and yield 12, 8, and 3 CDs respectively. Since everyone's share was 24/23 times larger than that claimed, everyone was happy, including the teacher who retrieved his Toccata and Fugue.

❾ Yes. By adding one CD and bringing the total to 12, we can give shares of 6, 3 and 2 CDs. The extra CD is then returned. Essentially we have reassigned the unallocated 1/12 share by increasing each share by a factor of 12/11 to obtain the shares 6/11, 3/11, and 2/11.

❿ Merely replace the 23 CDs in the original problem by 17 CDs.

The scoring guide presented below has been developed using student responses on a field test conducted in 1999. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 3

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Using Reasoning to Resolve Paradoxes Involving Fractions** (exercises ❶ – ❿) N 8-21, 8-22, 8-23	• Does not understand Kevin's point of view and/or is unable to divide the 24 cans according to both viewpoints. and • Is unable to determine the integral number of CDs in each share when the teacher's method is applied to 11 and to 23 CDs.	• Understands both Karen's and Kevin's points of view and is able to divide the 24 cans according to each viewpoint. or • Is able to determine the integral number of CDs in each share when the teacher's method is applied to 11 and to 23 CDs.	• Understands both Karen's and Kevin's points of view and is able to divide the 24 cans according to each viewpoint. and • Is able to determine the integral number of CDs in each share when the teacher's method is applied to 11 and to 23 CDs.	• Understands that the entitlements of Karen and Kevin are based on their contributions to the third party and not on their total contributions. and • Is able to create a problem that can be resolved by the method in the cartoon. (e.g., using 17 CDs)

The scoring guide presented above is a "product-oriented" rubric for assessing problem-solving skills based on the students' written work. However, this activity involves mainly group discussions of the ideas involved and therefore provides an excellent opportunity for observation-based assessment of students' problem-solving skills. To perform such an assessment you may wish to create your own "process-oriented" scoring guide. To help you get started, we have included an example of such a scoring guide for problem solving on page 89. However, as we have noted in the preambles to our previous scoring guides, it is recommended that you regard this merely as a starting point in the development of your own rubric that will evolve as you use it with students.

WHAT YOU MIGHT SEE

Problem Solving: Uses Reasoning to Resolve Paradoxes involving Fractions

Level 1

❶ a) What was the total cost of the pizza and soft drinks including tip and tax $24
b) How much was contributed by Kevin? by Karen? 15, 9
c) How many cans of soft drink are to be shared by Karen and Kevin? 15, 9

❷ Explain how Karen might justify her claim that she is entitled to 9 of the 24 cans of pop. She is to pay $9

❸ Explain how Kevin might justify his claim that he is entitled to 21 of the 24 cans of pop. He is just greedy. ← The student does not understand Kevin's point of view.

❹ How many cans of pop would you give to Kevin and Karen if you had to decide how to share the 24 cans between them? Justify your answer. Use a diagram and/or a logical argument to explain your thinking. 8 ← The student gives only one number and provides no explanation.

❺ Suppose Kevin had contributed $16 and Karen had contributed $8.
a) How many of the 24 cans of pop would Karen expect to receive? 8
b) How many of the 24 cans of pop would Kevin expect to receive? 16 ← The student has not understand Kevin's reasoning.
c) How many of the 24 cans would you give to each of them? Justify your answer. 8

❻ a) How many CDs did Kevin, Karen, and Conner win? 23
b) What share of the CDs did each of them claim? Kevin-12, Karen-8, Conner-3
c) Why were they unable to receive those shares? because 23 is an odd number
d) What did the music teacher do to resolve the problem? He gave 1 cd.

❼ a) Assuming each of the students received the shares they claimed, how many CDs did each of them receive? 12 8, 3 ← The correct number of CDs in each share is given.
b) How many CDs did the students receive in total? 23
c) Did the teacher's contribution provide shares of the 23 CDs thatwere different from the shares claimed by the students? If so, how did the shares change? He was knew he would get his CD back

❽ Explain why the teacher was able to divide the 23 CDs to everyone's satisfaction even though he did not really change the number of CDs to be divided.

❾ Suppose the students had won 11 CDs and claimed shares of 1/2, 1/4 and 1/6. Can you use the teacher's method to share the CD's to everyone's satisfaction? Explain why or why not. No, One person would recieave 1 and a half CD. ← The correct number of CDs in each share is not given because the student seems not to have understood the question.

❿ Create a problem like the one above that can be solved by the teacher's method. Assume that three people are to share some number of identical items and they claim shares of 1/2, 1/6, and 2/9.

Level 2

The student has understood both Kevin and Karen's points of view.

❶ a) What was the total cost of the pizza and soft drinks including tip and tax? 24$
b) How much was contributed by Kevin? by Karen? 15 9
c) How many cans of soft drink are to be shared by Karen and Kevin? 24

❷ Explain how Karen might justify her claim that she is entitled to 9 of the 24 cans of pop. 1 can per dollar paid

❸ Explain how Kevin might justify his claim that he is entitled to 21 of the 24 cans of pop. He paid 7 dollars extra where as karen only paid 1$ extra
7x3=21 cans 1x3=3 cans ← The correct division of cans according to each viewpoint.

❹ How many cans of pop would you give to Kevin and Karen if you had to decide how to share the 24 cans between them? Justify your answer. Use a diagram and/or a logical argument to explain your thinking. I would tell them to deal with it and hurry up cause I don't want to wait, besides Kevin was fine w/ Karen's soulution

❺ Suppose Kevin had contributed $16 and Karen had contributed $8.
a) How many of the 24 cans of pop would Karen expect to receive?
b) How many of the 24 cans of pop would Kevin expect to receive?
c) How many of the 24 cans would you give to each of them? Justify your answer.

None by Kevins way & 8 by her way

All of them his way & 16 Karen's way

I would keep out of it - let them handle it themselves - I don't like mental math

The student shows a political wisdom beyond her years.

78

In Exercises ❻ – ❿, the student was unable to demonstrate an understanding of the teacher's method of resolving the paradox and therefore was unable to determine the number of CDs in each share when applied to 11CDs. Therefore these responses were assessed at Level 2 rather than Level 3.

WHAT YOU MIGHT SEE

Problem Solving: Uses Reasoning to Resolve Paradoxes involving Fractions

Level 4

1.
a) The total cost was $ 24.00
b) Kevin contributed 62.5 %. Karen contributed 37.5 %. $15, $ 9
c) 24 cans

2. The total cost was $ 24.00, there are 24 cans. So Karen contributed $ 9 out of $24, so she gets 9 of 24 cans. Kevin contributed $15 out of $24, so he gets 15 of 24 cans.

← The student has understood Karen's point of view.

3. Kevin counted how much he and Karen contributed to the other girl

← The student has understood Kevin's point of view.

4. It would be pretty hard if I wanted it to be precise. I would calculate the cost of each can and then count how much cans would equal $1 and $7.

← The student has gone beyond expectations and sought to determine value of 24 cans of pop and then distribute that against the debt in the 7:1 ratio.

5.
a) Karen would expect to recieve $8
b) Kevin would expect to recieve all of the cans
c) I would give all of the cans to Kevin because he paid $8 for the other girl.

← Both points of view are understood and there is an appropriate division of the 24 cans using Kevin's reasoning.

6
a) They won 23 CD's
b) Kevin claimed a half, Karen claimed a third and Connor claimed an eighth
c) Because 23 can not divide by anything except itself and 1
d) The music teacher gave them 1 more CD to make it 24 CDs which is devisible by 2, 3 and 8

7.
a) Kevin got 12, Karen got 8 and Connor got 3
b) the students receive 24 in total.
c) The shares did change but they changed by a decimal number. Otherwise the students couldn't have agreed on a decision because you cannot split a CD in half

← The correct number of CDs in each share is given for the distribution of the 23 CDs.

← The student astutely observes that the shares changed "by a decimal number." That is, the teacher's method assigns the unallocated 1/24 so that each share of the 23 CDs is an integer.

8. He added one more so that the number of CD's can be devisible by all the numbers the students wanted

9. Yes, you can add one more in to make it 12, wich is devisible by 4, 2 and 6

← The student has recognized that adding 1 to 11 yields a number that is divisible by 2, 4, and 6 and can be distributed in integral shares.

10.
Karen, Kevin and Conner won $ 53 dollars in a trivia quiz contest. Karen answered ½ of the questions so she claimed a half of the $53. Kevin had answered 1/6 of the questions so he claimed 1/6 of the $53. Colling had answered 2/9 of the questions so he claimed 2/9 of the $ 53. The math teacher added $1 to solve their problem

← The student has discovered that 54 is divisible by 2, 6, and 9 and has used this fact to create a problem that can be resolved by Mr. Scully's method.

Activity 4 – Teacher Edition
What Fraction of the Numbers in Pascal's Triangle Are Even?

Expectations Addressed

N 8-5 explain numerical information in their own words and respond to numerical information in a variety of media.

N 8-6 demonstrate an understanding of operations with fractions.

N 8-8 express repeated multiplication as powers.

N 8-9 add, subtract, multiply, and divide simple fractions.

N 8-10 understand the order of operations with brackets and exponents and apply the order of operations in evaluating expressions that involve fractions.

N 8-11 apply the order of operations (up to three operations) in evaluating expressions that involve fractions.

N 8-22 explain the process used and any conclusions reached in problem solving and investigations.

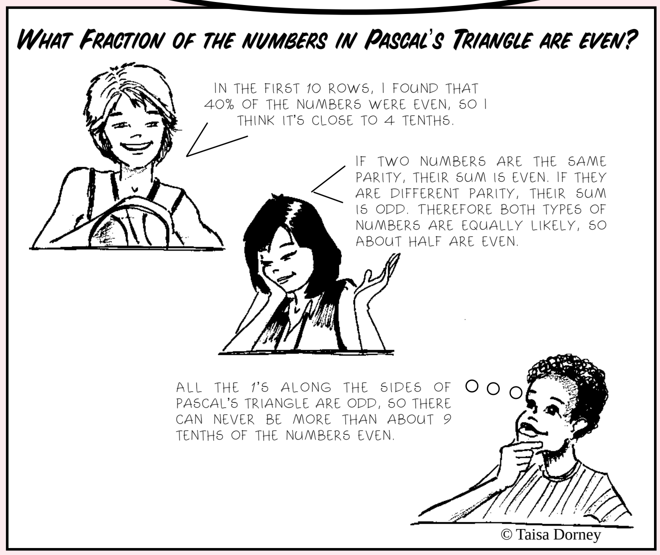
N 8-23 reflect on learning experiences and evaluate mathematical issues using appropriate mathematical language (e.g., in a math journal).

P&A 8-4 use the concept of variable to write equations and algebraic expressions.

Context

In this activity, students revisit Pascal's triangle (see the module, *Patterning & Algebra*.) They complete the first 16 rows of Pascal's triangle and record the fraction of numbers that are even. On that basis, they are asked to conjecture what fraction of the numbers in the first 256 rows of Pascal's triangle are even. Three possible fractions, 4/10, 1/2, and 9/10 are proposed and the students are asked to choose which of these they believe to be closest to the true fraction.

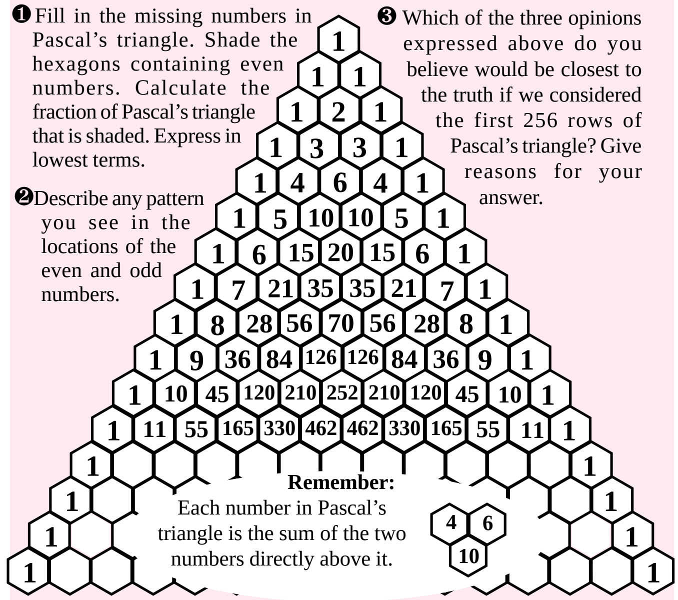
Activity 4 – Student Page



❶ Fill in the missing numbers in Pascal's triangle. Shade the hexagons containing even numbers. Calculate the fraction of Pascal's triangle that is shaded. Express in lowest terms.

❷ Describe any pattern you see in the locations of the even and odd numbers.

❸ Which of the three opinions expressed above do you believe would be closest to the truth if we considered the first 256 rows of Pascal's triangle? Give reasons for your answer.



On page 87, the students explore the procedure for generating the *Sierpinski triangle* – a famous example of the recently-discovered mathematical sets called a *fractals*. (Students will not need to know anything about fractals to complete the exercises.) In Exercises ❹ through ❻, students calculate the fraction of a triangle that remains each time one-quarter of it is removed. In Exercise ❼, they write an algebraic expression for the fraction that remains when this process is repeated n times. As the procedure is repeated a large number of times, students observe that the remaining area approaches zero. In the section titled *Research Report*, students shade in the *Pascal Parity Game Template* (see p. 92) the cells in Pascal's triangle that would contain even numbers using the fact that two numbers of the same parity (i.e. both even or both odd) have an even sum. As they apply this rule to shading the cells in the template, students are engaged in an exciting new topic in mathematics (invented by John von Neumann) called *cellular automata*. By comparing the pattern of shaded cells to the shaded parts of the Sierpinski triangle, students are guided to the conjecture that as the number of rows of Pascal's triangle increases, the proportion of even numbers approaches (but never reaches) 100%.

The Lesson Launch
5 minutes

Display on the overhead projector a copy of Pascal's triangle as shown on page 86. Review with students the procedure for generating the numbers in Pascal's triangle by forming two diagonal lines of 1's and then writing in any cell the sum of the two numbers directly above it. Have a student come to the overhead projector and fill in the numbers in the next row. Invite another student to shade the even numbers in Pascal's triangle.

Then ask students: *What percentage of the numbers are even: in the first 4 rows? in the first 8 rows? in the first 12 rows?*

Individual Activity
20 minutes

Distribute copies of page 86 to all students. Read the conjectures of the three students shown on that page regarding the fraction of numbers in Pascal's triangle that are even. Ask students which conjecture they believe is closest to the truth. Encourage students to give reasons for their conjectures. Then ask them to fill in the empty cells in Pascal's triangle on page 86 and use it to complete Exercises ❶ – ❸ on that page.

As you circulate around the room, check that students are recording a number close to 2/5 as the fraction of even numbers in their shaded triangles.

Paired Activity
30 minutes

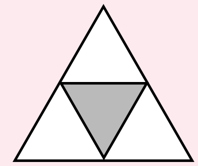
Distribute copies of page 87 to all students. Group the students into pairs. Assign Exercises ❹ – ❼ on that page. Before the students begin work, explain that they should think of the shaded areas as parts that are cut out of the original triangle. Explain also that on each repetition of the procedure, 1/4 of each remaining (white) triangle in the diagram is removed. Exercise ❼ is marked as a challenge because it requires students to generalize the procedure and to represent the remaining fraction of the original triangle after the n^{th} repetition as an algebraic expression in n, i.e., $(3/4)^n$. As you circulate around the classroom, it will be necessary to help some students discover that each time the procedure is applied, the area remaining is 3/4 the area before the procedure was applied. When all students have finished, discuss the answers to these exercises. Then distribute the *Pascal Parity Game* template to all students (see p. 92) and assign the *Research Report* activity.

Activity 4 – Student Page

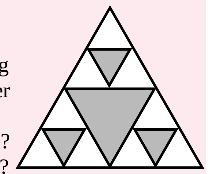
What Fraction of the numbers in Pascal's Triangle are even?

A pattern called *Sierpinski's gasket* is created as follows:

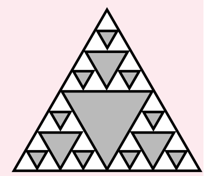
4 The midpoint of the sides of an equilateral triangle are joined to divide the original triangle into four smaller congruent triangles. The shaded triangle is then removed.
a) How many (white) congruent triangles remain?
b) What fraction of the original triangle remains?



5 The procedure above is then repeated, dividing each of the remaining triangles into four smaller triangles and removing the shaded triangles.
a) How many (white) congruent triangles remain?
b) What fraction of the original triangle remains?



6 Again, the procedure above is repeated, and the shaded triangles are removed.
a) How many (white) congruent triangles remain?
b) What fraction of the original triangle remains?



Challenge

7 Suppose the procedure above were repeated one more time. Write an expression for the fraction of the original triangle that would remain. Write an expression for the fraction of the original triangle that would remain if the procedure were repeated n times.

Research Report

Get the *Pascal Parity Game* template from your teacher. Follow the instructions on your template. Use your completed template to write a report that answers the question *What fraction of the numbers in Pascal's triangle are even?* Include these elements in your report:

- the patterns you obtained when you shaded the even numbers
- the number of hexagons that were shaded
- the total number of hexagons
- the fraction of the area that was shaded
- how the fraction of even numbers changed as the number of rows increased
- your evaluations of the opinions expressed on the other page

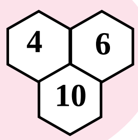
8 If you have a graphing calculator, get program PASCAL from your teacher. Run PASCAL to determine the percentage of even numbers in the first:
a) 16 rows b) 32 rows c) 64 rows
of Pascal's triangle.

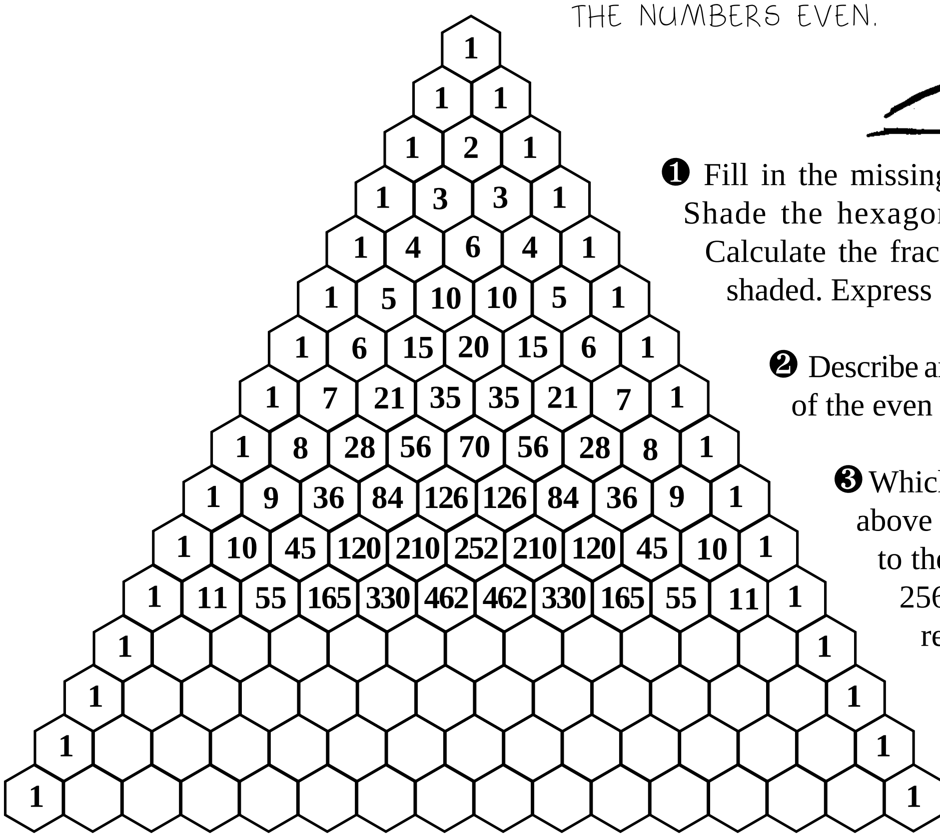
Closure

Collect the student reports. Then display on the overhead projector a transparency showing the completed Pascal Parity Game template shown on page 93. Discuss how this pattern resembles the Sierpinski triangle. Ask students what this suggests about the fraction of cells in Pascal's triangle that are shaded. Through questioning, guide students to the realization that as the number of rows of Pascal's triangle increases, the proportion of even numbers approaches 100%. If you have a TI-83 graphing calculator, input the program given on page 93 and display the shaded Pascal's triangle for various numbers of rows. If students have access to these graphing calculators, use the link connector to transfer this program and assign Exercise ❽.

WHAT FRACTION OF THE NUMBERS IN PASCAL'S TRIANGLE ARE EVEN?

Remember:
Each number in Pascal's triangle is the sum of the two numbers directly above it.





❶ Fill in the missing numbers in Pascal's triangle. Shade the hexagons containing even numbers. Calculate the fraction of Pascal's triangle that is shaded. Express in lowest terms.

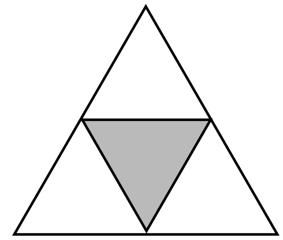
❷ Describe any pattern you see in the locations of the even and odd numbers.

❸ Which of the three opinions expressed above do you believe would be closest to the truth if we considered the first 256 rows of Pascal's triangle? Give reasons for your answer.

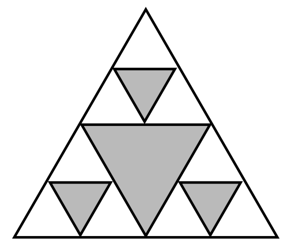
WHAT FRACTION OF THE NUMBERS IN PASCAL'S TRIANGLE ARE EVEN?

A pattern called *Sierpinski's gasket* is created as follows:

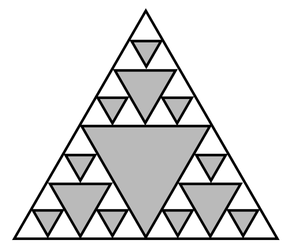
4 The midpoint of the sides of an equilateral triangle are joined to divide the original triangle into four smaller congruent triangles. The shaded triangle is then removed.
a) How many (white) congruent triangles remain?
b) What fraction of the original triangle remains?



5 The procedure above is then repeated, dividing each of the remaining triangles into four smaller triangles and removing the shaded triangles.
a) How many (white) congruent triangles remain?
b) What fraction of the original triangle remains?



6 Again, the procedure above is repeated, and the shaded triangles are removed.
a) How many (white) congruent triangles remain?
b) What fraction of the original triangle remains?



CHALLENGE

7 Suppose the procedure above were repeated one more time. Write an expression for the fraction of the original triangle that would remain. Write an expression for the fraction of the original triangle that would remain if the procedure were repeated *n* times.

RESEARCH REPORT

Get the *Pascal Parity Game* template from your teacher. Follow the instructions on your template. Use your completed template to write a report that answers the question *What fraction of the numbers in Pascal's triangle are even?* Include these elements in your report:

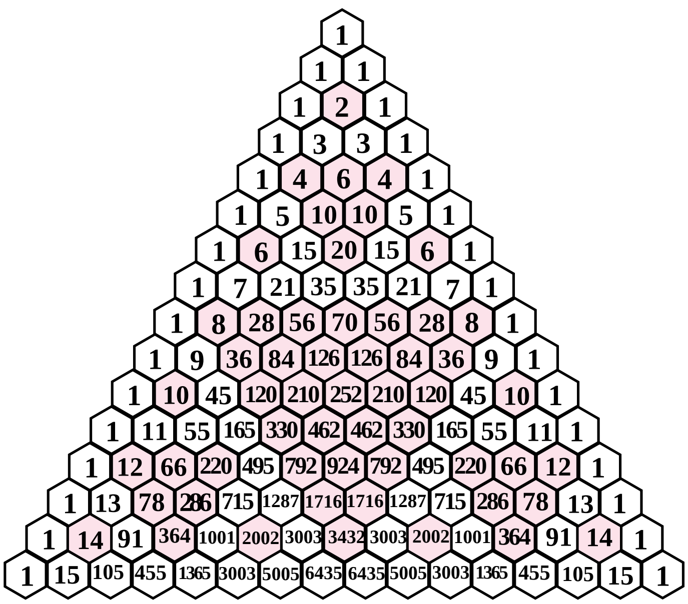
- the patterns you obtained when you shaded the even numbers
- the number of hexagons that were shaded
- the total number of hexagons
- the fraction of the area that was shaded
- how the fraction of even numbers changed as the number of rows increased
- your evaluations of the opinions expressed on the other page

8 If you have a graphing calculator, get program PASCAL from your teacher. Run PASCAL to determine the percentage of even numbers in the first:
a) 16 rows b) 32 rows c) 64 rows
of Pascal's triangle.

Grade 8

Answer Key for Activity 4

❶ The first 16 rows of Pascal's triangle are shown below. The cells containing even numbers are coloured.



There are 55 coloured cells out of 136 cells, so the fraction of cells that are coloured is $55/136 \approx 0.404$.

❷ We see from the diagram above that the even numbers form a large equilateral triangle at the centre of Pascal's triangle together with 3 small equilateral triangles near the corners of Pascal's triangle. Furthermore, every other cell along the internal border (inside the 1's) is an even number.

❸ At this point, most students will agree with one of the first two students on page 86, although the even numbers will eventually dominate and approach 100% of all the numbers in Pascal's triangle. There are 26 335 even numbers out of 32 896 numbers in the first 256 rows of Pascal's triangle (slightly more than 80%), so the 9 tenths estimate is closest.

❹ a) 3 white triangles remain.
b) 3/4 of the original triangle (by area) remains.

❺ a) 9 white triangles remain.
b) $(3/4)^2$ or 9/16 of the original triangle remains.

❻ a) 27 white triangles remain.
b) $(3/4)^3$ or 27/64 of the original triangle remains.

❼ 81 white triangles would remain. Each triangle would be 1/256 of the original triangle, so the fraction of the original triangle that would remain is $(3/4)^4$ or 81/256. Each time the process is repeated, the number of triangles remaining is tripled but the areas of these triangles is one quarter their previous size. Therefore their total area is 3/4 the area prior to the application of the procedure. After n applications of the procedure, the area remaining is $(3/4)^n$ times the area of the original triangle. Clearly as n becomes arbitrarily large, the unshaded area approaches 0. The shaded area approaches the area of the original triangle.

The set defined by this process is known as the *Sierpinski gasket* or *Sierpinski triangle*.

❽ Using program PASCAL, we obtain these results.

a) The number of even numbers in the first 16 rows of Pascal's triangle is 55. This checks with our answer in Exercise ❶. This corresponds to about 40.4% of all the numbers.

b) Proceeding as in ❽ a, we obtain 285 even numbers in the first 32 rows of Pascal's triangle. This corresponds to almost 54% of all the numbers.

c) Proceeding as in ❽ a and b, we obtain 1351 even numbers in the first 64 rows of Pascal's triangle. This is about 65% of all the numbers.

Research Report

The pattern that should result from shading is shown on page 93. There are a total of 285 shaded cells out of 528 cells, so 285/528 or 53.9% of the numbers in the first 32 rows of Pascal's triangle are even.

In general, the number of even numbers in Pascal's triangle, up to and including row 2^n, is given by:

$$(2^{n-1}-1)\bullet 2^{n-2} + (2^{n-2}-1)\bullet 2^{n-3}\bullet 3 + (2^{n-3}-1)\bullet 2^{n-4}\bullet 3^2 + \dots + (2^2-1)\bullet 2\bullet 3^{n-3} + 3^{n-2}$$

Substituting $n = 5$ and 6 yields 285 and 1351 even numbers as indicated in ❽ above. This formula can be used to show that the proportion of even numbers in the first 2^n rows of Pascal's triangle approaches 100% as n increases.

A "Process-Oriented" Scoring Guide

The scoring guide presented below is a "process-oriented" scoring guide to be used to assess problem solving skills as the students work through Activity 4. By contrast, the scoring guides in the previous activities are "product-oriented" and are intended for use in the assessment of completed student work. The reason for including a process-oriented scoring guide in this unit is to remind you that assessment should include more than the evaluation of a "written product" and ought to include your informal observations of students as they work alone or in groups. Informal conversations with students about the concepts involved in an activity can often elicit greater insights into the student's thinking processes than what is recorded on a page.

However, as we have noted in the preambles to our previous scoring guides, it is recommended that you regard this merely as a starting point in the development of your own scoring guide that will evolve as you use it with students.

Scoring Guide for Activity 4

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Discover a Pattern & Generalize to Test a Conjecture.** N 8-23, P&A 8-4	• Does not understand the question, "What fraction of the numbers in Pascal's triangle are even?" • Needs assistance in identifying the cells to be shaded in Pascal's triangle. • Has significant difficulty writing fractions for the areas in the generation of the Sierpinski gasket. • Unable to see a relationship between the shaded triangles in the Sierpinski gasket and the even numbers in Pascal's triangle.	• Understands the question, "What fraction of the numbers in Pascal's triangle are even?" • Is able to identify the cells to be shaded in Pascal's triangle. • Makes one or two errors in writing fractions for the areas in the generation of the Sierpinski gasket. • Understands that the shaded triangles in the Sierpinski gasket have a similar pattern to the even numbers in Pascal's triangle.	• Understands the question, "What fraction of the numbers in Pascal's triangle are even?" • Is able to identify the cells to be shaded in Pascal's triangle. • Makes no errors in writing fractions for the areas in the first three stages in the generation of the Sierpinski gasket. • Understands that the shaded triangles in the Sierpinski gasket have a similar pattern to the even numbers in Pascal's triangle. • Understands that the percentage of even numbers in Pascal's triangle "tends to" increase as the number of rows increases.	In addition to Level 3: • Is able to write an expression for the fraction of the original triangle remaining in the n^{th} stage of the generation of the Sierpinski gasket. • States which of the conjectures presented on page 86 is most reasonable and supports this statement with reference to the increasing percentages of even numbers in Pascal's triangle as the number of rows is increased.

WHAT YOU MIGHT SEE

Problem Solving: Discover a Pattern and Generalize to Test a Conjecture

Samples of the Written Work

Sample 1

Pascal's Triangle
4. a) 3 b) 3/4
5. a) 9 b) 9/13
6 a) 27 b) 27/40
7. 81

← This work shows that the student was able to obtain the correct numerators, but the denominators are incorrect. A brief interview would clarify whether the student was making a simple counting error or whether the process was fundamentally flawed.

Sample 2

2 2nd inner line +1.

← The pattern identified by this student is that the numbers in the second diagonal line increase by 1 as you travel downward.

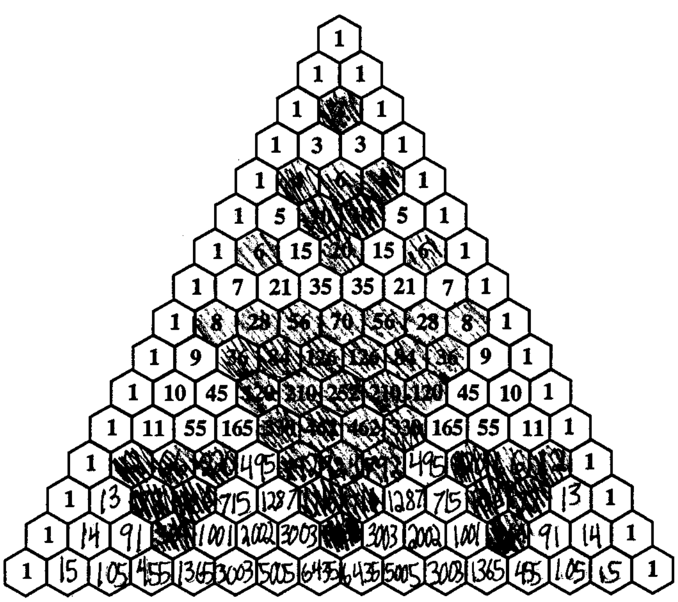
3 The opinion with Is on the sides of the pascal because it is true.

← The student agrees with Kevin's conjecture, because Kevin has indicated that 1's fall along the outside lines of Pascal's triangle. It would be informative to question the student further.

4 a) 3 5 a) 9 6 a) 27 7 a) $n3$
b) 3/4 b) 1/2 b) 1/4

↑The student has written n^3 instead of $(3/4)^n$. The student might be on the right track.

Sample 3



← The completed template shows that the student is able to add correctly and is usually able to distinguish between even and odd numbers, although he missed shading a few even numbers.

WHAT YOU MIGHT SEE

Problem Solving: Discover a Pattern and Generalize to Test a Conjecture

Samples of the Written Work

Sample 4

What Fraction of the Numbers in Pascal's Triangle are Even?

A pattern called *Sierpinski's gasket*, is created as follows:

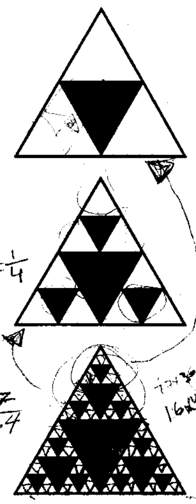
4 The midpoint of the sides of an equilateral triangle are joined to divide the original triangle into four smaller congruent triangles. The shaded triangle is then removed.
a) How many (white) congruent triangles remain? 3
b) What fraction of the original triangle remains? 3/4

5 The procedure above is then repeated, dividing each of the remaining triangles into four smaller triangles and removing the shaded triangles.
a) How many (white) congruent triangles remain? 9
b) What fraction of the original triangle remains? 9/16

3 x 3 = 9
4 x 4 = 16

6 Again, the procedure above is repeated, and the shaded triangles are removed.
a) How many (white) congruent triangles remain? 27
b) What fraction of the original triangle remains? 27/64

9 x 3 = 27
4 x 16 = 64



81
16 x 4 = 64

CHALLENGE **7** Suppose the procedure above were repeated one more time. Write an expression for the fraction of the original triangle that would remain. Write an expression for the fraction of the original triangle that would remain if the procedure were repeated *n* times.

What # is that

81 Congruent Δ & 81/256 Fraction Δ ←

This work shows that the student was able to obtain the correct fractions for the first four stages of the development of the Sierpinski gasket. The comment in Exercise **7** indicates that the student was unfamiliar with the algebraic notation for the general case. This would be an excellent occasion to explain its meaning and see whether the student could develop the expression $(3/4)^n$ and then deduce its value as *n* gets large.

Sample 5

Act #4 What Fraction of the number of Pascal's Triangle are even?

1. on sheet
2. The even numbers appere in triangles it becomes larger as the triangle grows
3. the second

Act 4
(4) a) 3 white triangles remain
b) 3/4

(5) a) 9 white triangles remain
b) 9/16

(6) a) 27 white triangles remain
27/64

(7) 81/256 multiple amout of white by 3 and the whole triangle by 4. Because the fraction is always 3/4

$n = \frac{3}{4} \times n$

16 x 4 = 64; 27 x 3 = 81; 64 x 4 = 256

(8) a) 40.9% even

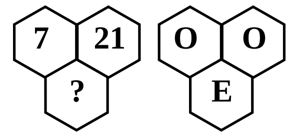
This student was able to obtain the correct fractions for the first four stages of the development of the Sierpinski gasket. Furthermore, the student has discovered that the fraction remaining at each stage is 3/4 the fraction remaining in the previous stage. A brief interview would reveal whether the student is able to make a conjecture about the fraction of numbers that are even.

TEMPLATE

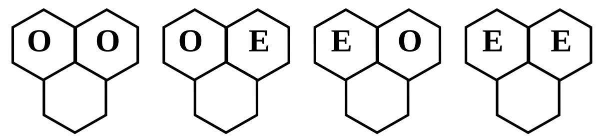
The Pascal Parity Game

In Activity 4, you completed the first 16 rows of Pascal's triangle and then shaded those hexagons containing even numbers. In this activity, you will shade the even numbers in the first 32 rows of Pascal's triangle *without any computation*!

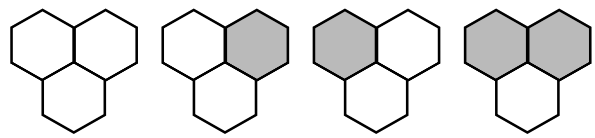
Each number in Pascal's triangle is the sum of the two numbers immediately above it. To determine whether the missing number in the diagram on the right is even or odd, we could just add 7 + 21 to obtain 28. Alternatively, you could observe that 7 and 21 are odd so their sum is even. We show this by writing O in hexagons (cells) containing odd numbers and E in cells containing even numbers.



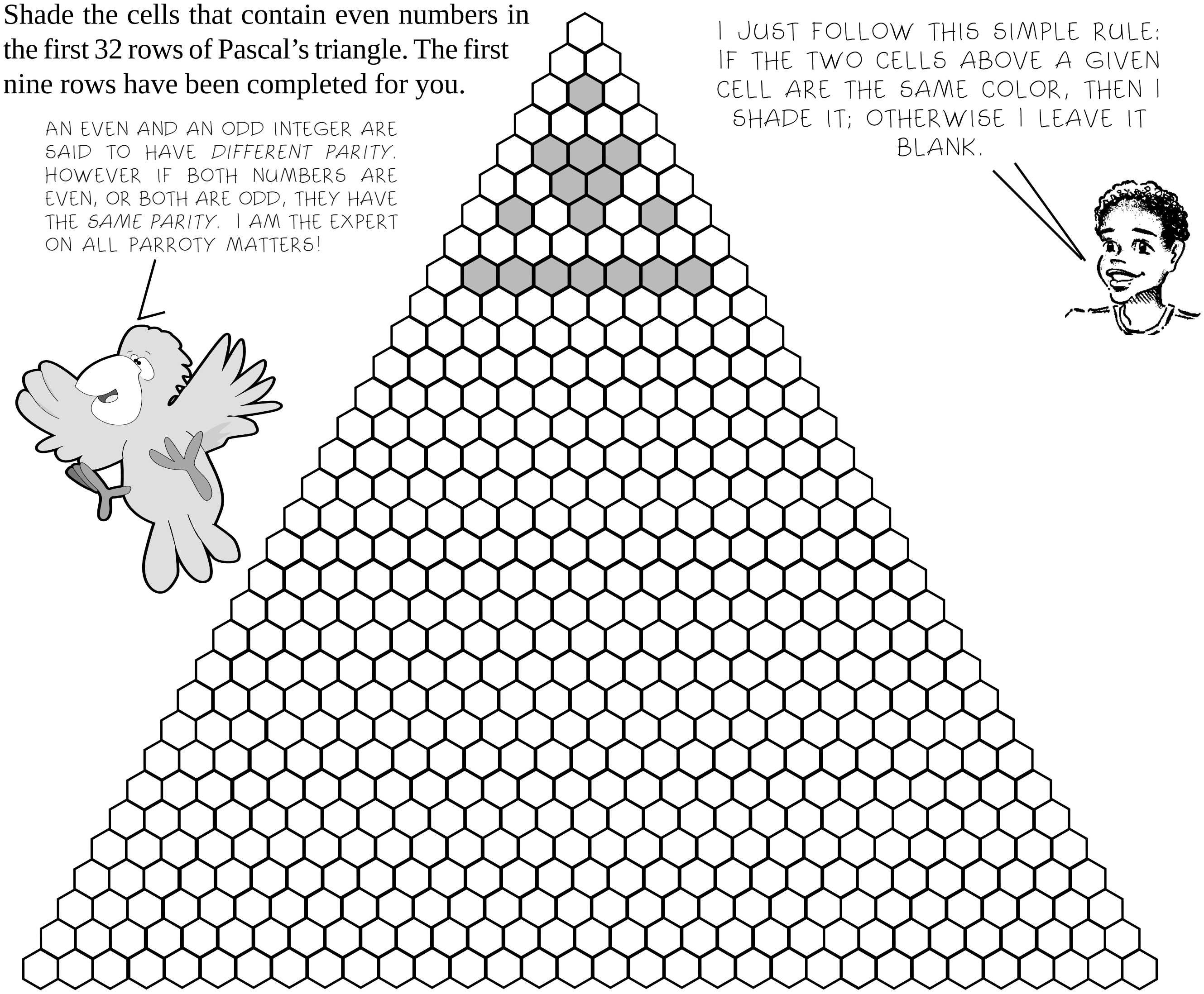
Write O (for odd) or E (for even) in each empty cell.



Instead of writing O or E in each cell, we can save time by shading the cells containing even numbers and leaving blank the cells containing odd numbers. Shade or leave blank the bottom cell in each diagram to indicate whether the number in it is odd or even.

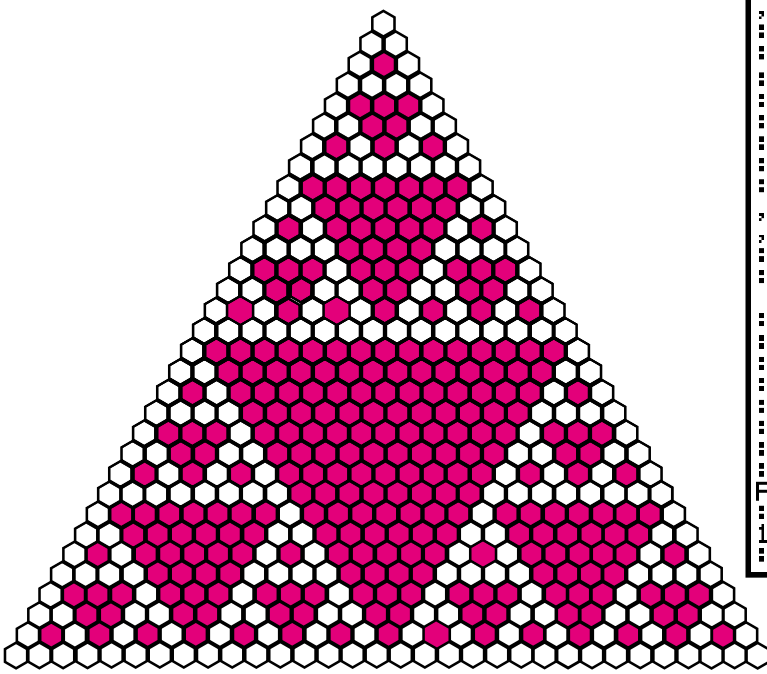


Shade the cells that contain even numbers in the first 32 rows of Pascal's triangle. The first nine rows have been completed for you.



Answers to The Pascal Parity Game

The diagram shows the completed Pascal Parity Game. There are 32 rows containing 528 cells. Of these, 285 cells are shaded so the fraction of even numbers in the first 32 rows of Pascal's triangle is 285/528 or about 54%. On a graphing calculator such as a TI-83, you can display more than 32 rows by using program PASCAL shown here.



```
PROGRAM:PASCAL
:ClrDraw
:Input N
:1→E
:Pxl-On(1,47)
:For(I,4,N,2)
:47-(I/2)→Z
:For(K,1,I-2)
:Z+K→L
:If pxl-Test(I-3
,L)=pxl-Test(I-3
,L+1)
:Then
:Pxl-On(I-2,L)
:E+1→E
:End
:End
:For(K,0,I-2)
:Z+K→L
:If pxl-Test(I-2
,L)=pxl-Test(I-2
,L+1) and I<N
:Then
:Pxl-On(I-1,L+1)

:E+1→E
:End
:End
:End
:Pause
:Disp E
:Pause
:Disp "PERCENT O
F EVENS IS"
:Disp E/((N)*(N+
1)/2)*100
:
```

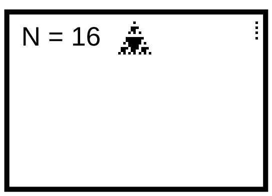
Teacher Note:

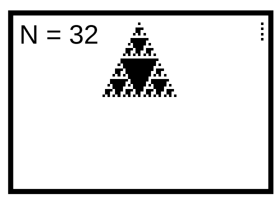
To run PASCAL input the number of rows, N, and wait for the pause cursor. Then press the ENTER key to get the number of even numbers. Press ENTER again to get the percentage of even numbers.

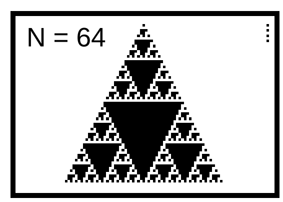
The displays below show the results obtained for N = 16, 32, and 64. The program PASCAL will display an error message when values of N larger than 64 are input because this is the number of rows on the screen of the TI-83 .

The displays show that "complete" pictures of the "Sierpinski triangle" are displayed when N is a power of 2, that is, when $N = 2^n$ for some value of n.

We observe that the percentage of even numbers in Pascal's triangle increases from 40% to 54% and to 65% as the number of rows increases from 16 to 32 and to 64 respectively. The table on the right shows that in the first 128 rows of Pascal's triangle [i.e., $N = 2^n$ for $n = 7$] there are 6069 even numbers [u(7) = 6069] out of 8256 numbers [v(7) = 8256], so the percentage of even numbers is about 74%. The first 256 rows contain 26 335 even numbers out of 32 896 numbers, so the percent of even numbers is about 80%.








```
?16
Number of even numbers →            55
PERCENT OF EVEN…
                       40.44117647
                              Done
```

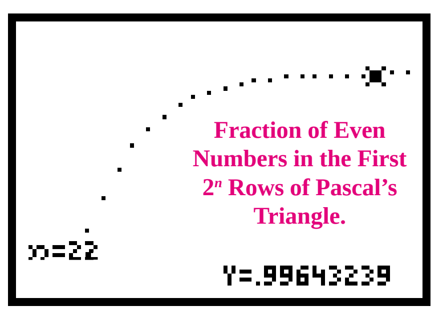
```
?32
Number of even numbers →           285
PERCENT OF EVEN…
                       53.97727273
                              Done
```

```
?64
Number of even numbers →          1351
PERCENT OF EVEN…
                       64.95192308
                              Done
```

n	$u(n)$	$v(n)$
2	1	10
3	9	36
4	55	136
5	285	528
6	1351	2080
7	6069	8256
8	26335	32896
9	111645	131328
10	465751	524800
11	1.92E6	2.1E6
12	7.86E6	8.39E6
13	3.2E7	3.36E7
14	1.29E8	1.34E8
15	5.23E8	5.37E8

n=15

If we define w(n) to be the fraction of even numbers in the first 2^n rows of Pascal's triangle, then the graph of w(n) vs. n in the display on the right shows that w(n) approaches 1 or 100% as n increases. By tracing along the graph we see that in the first 2^{22} rows of Pascal's triangle, the percentage of even numbers exceeds 99.6%.



This is for your information only.

TEMPLATE

Record of Student Achievement on the Grade 7 Unit

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Problem Solving	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Communication	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Concepts	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Problem Solving	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Communication	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Concepts	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Problem Solving	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Communication	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Concepts	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Problem Solving	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Communication	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Concepts	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 27

Topic	Level
Problem Solving	

From Scoring Guide for Activity 2 p. 35

Topic	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 3 p. 43

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 51

Topic	Level
Communication	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Concepts	
Communication	

TEMPLATE

Record of Student Achievement on the Grade 8 Unit

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Application	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Application	
Communication	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Problem Solving	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Application	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Application	
Communication	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Problem Solving	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Application	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Application	
Communication	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Problem Solving	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Application	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Application	
Communication	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Problem Solving	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 1 p. 65

Topic	Level
Application	

From Scoring Guide for Activity 2 p. 73

Topic	Level
Application	
Communication	

From Scoring Guide for Activity 3 p. 81

Topic	Level
Problem Solving	

From Scoring Guide for Activity 4 p. 89

Topic	Level
Problem Solving	

Combining the Scores from all Scoring Guides

Topic	Level
Problem Solving	
Application	
Communication	

Additional Resources for Number Sense & Numeration

Beckmann, Petr. *A History of π.* New York, NY: St. Martin's Press, 1971.

Blatner, David. *The Joy of π.* New York, NY: Walker Publishing Company Inc., 1997.

Lemon, Patricia. "Pascal's Triangle–Patterns, Paths, and Plinko." *Mathematics Teacher*, Vol 90, #4. Reston, VA: National Council of Teachers of Mathematics, April, 1997, pp. 270-273.

McIntosh, Alistair, Robert E. Reys, and Barbara Reys. "Mental Computation in the Middle Grades: The Importance of Thinking Strategies." *Mathematics Teaching in the Middle School.* Vol 2, #5. Reston, VA: National Council of Teachers of Mathematics, March–April, 1997, pp. 322-327.

Mikusa, Michael. "Problem Solving is More Than Solving Problems." *Mathematics Teaching in the Middle School.* Vol 4, #1. Reston, VA: National Council of Teachers of Mathematics, September, 1998, pp. 20-25.

Mingus Tabitha T.Y. and Richard M. Grassl. "Algorithmic and Recursive Thinking: Current Beliefs and Their Implications for the Future." In *The Teaching and Learning of Algorithms in School Mathematics.* Reston, VA: 1998 Yearbook of the National Council of Teachers of Mathematics.

Morrow, Lorna J. and Margaret J. Kenney, editors. *The Teaching and Learning of Algorithms in School Mathematics.* Reston, VA: 1998 Yearbook of the National Council of Teachers of Mathematics.

Paulos, John Allen. *Innumeracy: Mathematical Illiteracy and its Consequences.* New York, NY: Hill and Wang, 1988.

Reys, Barbara J. "Developing Number Sense in the Middle Grades." From *Addenda Series, Grades 5–8: Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics, 1992.

Ross, Joan and Marc Ross. "Fermi Problems: Or How to Make the Most of What You Already Know." In *Estimation and Mental Computation.* Reston, VA: 1986 Yearbook of the National Council of Teachers of Mathematics, pp. 175–181.

Steen, Lynn A. "Pattern." In *On the Shoulders of Giants: New Approaches to Numeracy,* edited by Lynn A. Steen. Washington D.C.: National Academy Press, 1990.

Tahan, Malba. *The Man Who Counted.* New York, NY: W.W. Norton & Company Inc., 1993.

Web Sites

Classic Fermi Questions	http://forum.swarthmore.edu/workshops/sum96/interdisc/classicfermi.html
CN Tower History	http://www.cntower.ca/l1_hist.html
The CN Tower Calculator	http://www.cntower.ca/L1_calc.html
The Fraser Institute	http://www.fraserinstitute.ca/fedbudg.htm
The Royal Canadian Mint	http://www.rcmint.ca/en/

Free Software for Ontario Schools

The Ministry of Education and Training of Ontario purchases site licences of software for all publically funded schools in the province. This software can be obtained from the Ontario Educational Software Service (OESS) representative in your school district. To determine what is available, access this web site: **http://www.tvo.org/osapac**